FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS

by

POONACHA P G

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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POONACHA P G

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Dedicated to
all those who feel bad about me at
least once in their life time

We say that the one and many become identified in our propositions and that now as in time past they run about together in every sentence which is uttered and that this union of them will never cease and is not now beginning but is as I believe an ev rlasting quality of propositions themselves which never grows old But any young man when he first tastes these subtleties is delighted and fancies that he has found a treasure of wisdom in the first enthusiasm of his joy he leaves no stone or rather no thought unturned now rolling up the many into the one and kneading them together now unfolding and dividing them he puzzles himself first and above all and then he proceeds to puzzle his neighbors whether they are older or younger or of his own age - that makes no difference neither fath r nor mother does he spare no human being who has ears is safe from him hardly even his dog and a barbarian would have no chance of escaping him if an interpreter could only be found

Socrates (The dialogues of Plato
Jowett vol III Philebus p 566)

CERTIFICATE

Certified that this work FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS by Mr P G Poonacha has been carried out under my supervision and that this has not been submitted elsewhere for a degree

October 1986

(M U SIDDIQI)
Department of Electrical Engineering
Indian Institute of Technology
Kanpur

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Poor Guy

TABLE OF CONTENTS

	LIST OF TABLES	x
	LIST OF FIGURES	xi
	LIST OF SYMBOLS AND ABBREVIATIONS	xiii
	SYNOPSIS	xvi
CHAPTER 1	INTRODUCTION	1
	1 1 Motivation	2
	1 2 Review of Literature	4
	1 2 1 Data Recovery Schemes for Continuous Serial Data Transmission	5
	1 2 2 Data Recovery Schemes for Block Data Transmission	8
	1 3 Key Results and Observations	9
	1 4 Organization of the Thesis	16
	1 5 Notations and Conventions	16
CHAPTER 2	BLOCK DATA TRANSMISSION	18
	2 1 Assumptions	18
	2 2 Discrete Time Model for ISI Channels	21
	2 3 Block Data Transmission Format	24
	2 4 The Deconvolution Problem	27
CHAPTER 3	MAXIMUM LIKELIHOOD DECONVOLUTION	32
	3 1 Recursive Algorithm for Minimization of Nonlinear Function of Several Variables	33
	3 1 1 Real Variable Case	38
	3 1 2 Discrete Variabl Case	40

Page

3 2 Recursive Algorithm for Banded Toeplitz Quadratic Functions 43

3 3 Viterbi Algorithm for Minimizing BTQF 46

3 4 Strategies for Reducing Computational and Storage Requirements 51

3 4 1 Binary Data Transmission 54

3 4 2 Minimization of Pseudo-Boolean Functions 56

3 5 Recursive Algorithm for Decoding Block Error Correcting Codes 59

CHAPTER 4 DECISION FEEDBACK DECONVOLUTION SCHEMES 70

4 1 Decision Feedback Scheme Using Cholesky Decomposition 72

4 2 MMSE Decision Feedback Data Recovery Scheme 77

4 3 Decision Feedback Scheme Based on Minimum Property of Fourier Expansions 81

4 4 Decision Feedback Scheme Based on Modified Austin s Approach 87

4 4 1 Computation of the Tap Coefficients 93

4 4 2 Modified Austin s Approach and Nonlinear Gaussian Elimination Method 94

4 5 Some Observation 96

CHAPTER 5 DATA RECOVERY SCHEMES USING k-CIRCULANT COMPLETIONS 98

5 1 k-Circulant Completion and Finite Deconvolution 99

5 1 1 Deconvolution Using Circulant Completion 101

5 1 2 Deconvolution using k-Circulant Completion 101

5 1 3 Need for Deconvolution by k-Circulant Completion 104

Page

	5 2 Deconvolution by Direct Inversion	105
	5 3 Data Recovery Schemes Based on k-Circulant Completions	111
	5 4 Performance Considerations	113
CHAPTER 6	LEAST SQUARES DECONVOLUTION VIA k-CIRCULANT COMPLETIONS	121
	6 1 Square Completion and Solution of Linear Equations	124
	6 2 Least Squares Minimum Norm Solution and Moore-Penrose Generalized Inverse	126
	6 3 Least Squares Solution Via Square Completions	127
	6 4 Channels Amenable to LS Solution Via k-Circulant Completions	130
	6 5 Characterization of Orthogonal k-Circulants	133
	6 6 LS Solution Using k-Circulant Completion	136
CHAPTER 7	LEAST SQUARES DECONVOLUTION - ITERATIVE APPROACH	142
	7 1 Review of Steepest Descent Algorithm	145
	7 2 Application of DFT to Steepest Descent Algorithm	146
	7 3 Derivation of the DFT Domain Steepest Descent Algorithm	148
	7 3 1 Data Recovery Scheme Based on SDA in the DFT Domain	153
	7 4 Rate of Convergence of SD	155
	7 5 Fast Converging Steepest Descent Algorithm	156
	7 6 Fixed Step Steepest Descent Algorithm	158

Page

CHAPTER 8 PERFORMANCE OF DATA RECOVERY SCHEMES 164

8.1 Good Channels for Block Data Transmission 165

8.1.1 Maximum Minimum Distance Channels 165

8.1.2 Maximum Distance Channels 170

8.1.3 Procedure for Obtaining MMD and MD Channels 171

8.1.4 Computation of Minimum Distance of a Channel 174

8.2 Channels Used in the Study 176

8.3 Simulation Studies on Data Recovery Schemes 177

8.3.1 The Effect of Increased ISI Among the Block Elements 191

CHAPTER 9 CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH 197

APPENDIX A WITTEN'S LEMMA 203

APPENDIX B k-CIRCULANT MATRICES 204

REFERENCES 209

LIST OF TABLES

Table No	Title	Page
3 1	Stage 1	41
3 2	Stage 2	41
3 3	Stage 1	44
3 4	Stage 2	45
3 5	Stage 3	45
3 6	Stage 4	45
3 7	Stage 5	46
3 8	Stage 6	46
8 1	Some examples of MD and MMD channels	173
8 2	Channels used in the simulation study	176
8 3	Channel sample response coefficients	193

LIST OF FIGURES

Figure No	Title	Page
2 1	Block diagram of data transmission system	19
2 2	Discrete model of finite impulse response channel	22
2 3	Response of a channel to CSDT and BDT in the absence of noise	26
2 4	Plot of g versus m	28
3 1	Trellis diagram showing survivor paths	50
3 2	Development of the tables in ML-RDA for xample 3 1	52
4 1	Block diagram of the data recovery scheme for BDT using Cholesky decomposition	75
4 2	Block diagram of the data recovery scheme using minimum property of Fourier xpansion	85
4 3	Block diagram of the data recovery scheme using modified Austin s approach	91
5 1	Block diagram of the data recovery scheme using k-DFT	112
5 2	Data recovery scheme based on k-Circulant completion	112
7 1	Data recovery scheme based on DFT domain steepest descent algorithm	154
8 1	Comparison of performance of various DR schemes	179
8 2	Comparison of performance of various DR schemes	180
8 3	Comparison of performance of various DR schemes	181
8 4	Comparison of performanc of various DR schemes	182
8 5	Comparison of performance of various DR schemes	183
8 6	Comparison of performance of various DR schemes	184
8 7	Comparison of performance o various DR schemes	185

Figure No	Title	Page
8 8	Comparison of performance of various DR sch nes	186
8 9	Comparison of performance of various DR schemes	187
8 10	Comparison of MLB and MLC performance	189
8 11	MMD channel performance	191
8 12	Effect of m on the performance of CCB and SCB	192
8 13	Effect of increased ISI on the performance of MLB	194
8 14	Effect of increased ISI on the performance of MAB	194
8 15	Effect of increased ISI	196
8 16	ect of increased ISI	196

LIST OF SYMBOLS AND ABBREVIATIONS

$\|\|A\|\|_e$	Euclidean norm of a matrix A
BDT	Block data transmission
BTQF	Banded Toeplitz quadratic function
CSDT	Continuous serial data transmission
DF	Decision feedback
DFE	Decision feedback equalizer
DFT	Discrete Fourier transform
DR	Data recovery
d_{min}	Minimum distance
FFT	Fast Fourier transform
$F(\underline{x})$	Function of n variables where $x = (x_1\ x_2\quad x_n)^T$
g	Number of ISI components
GE	Gaussian elimination
GF(2)	Galois field of two elements 0 and 1
H	Channel convolution matrix
H_k	k Circulant completion of H
h_i $o \leq i \leq g$	Channel parameters
H^{LS}	Least squares inverse of H
h(t)	Channel impulse response
H(z)	z-transform of the autocorrelation function of h(t)
H_1	Circulant completion of H
H_{-1}	Skew-Circulant completion of H
H^+	Moore Penrose inverse of H

ISI	Intersymbol interference
LS	Least squares
MD	Maximum distance
MMD	Maximum minimum distance
ML	Maximum likelihood
ML-RDA	Maximum likelihood recursive deconvolution algorithm
MSE	Mean square error
N_o	Noise variance
PAM	Pulse amplitude modulation
PBF	Pseudo Boolean function
$(PER)_i$	Probability of ith data symbol being in error
Pe	Probability of error
$Q(x)$	$= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy$
$r(A)$	Rank of a matrix A
SDA	Steepest descent algorithm
SNR	Signal-to-noise ratio
$\{S_k\}$	State sequence
T_c	Channel data rate in symbols per second
T_h	Channel impulse response duration
T_s	Source data rate in symbols per second
w_i	Sequence of independent Gaussian random variables with mean zero and variance $N_o/2$

VA	Viterbi algorithm
$\{x_i\}$	Sequence of statistically independent data symbols
$X(z)$	z-transform of the autocorrelation function of $h(t)$
$\{y_k\}$	Output sequence
η_k	ν-Circulant matrix with first row (0 0 0 k)

SYNOPSIS

Poonacha P G Ph D
Department of Elec rical Engineering
Irdian Institute of Technology Kanpur
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FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS

In this thesis we study several data recovery (DR) schemes for block data transmission (BDT) over finite impulse response channels in the presence of noise In block data transmission due to Clark the input sequence is partitioned into blocks which are separated by guard zones of appropriate duration such that the channel output blocks do not overlap Since the channel is assumed to be linear its output can be represented as convolution of the input data values and the channel impulse response The operation of recovering data from the nonoverlapping blocks at the receiver is then essentially that of finite deconvolution in the presence of noise

We formulate the deconvolution problem as minimization of a banded Toeplitz quadratic function with data as unknown variables When the unknown variables are real minimization of the quadratic function is straightforward However when the variables take values from a finite alphabet as in the case of data transmission it is difficult to provide computationally attractive solutions The solution to the deconvolution problem is

optimal or suboptimal depending on the assumptions made on the unknown variables and the strategy adopted for solving the problem

We study maximum likelihood decision feedback and least squares deconvolution methods in detail Among these schemes maximum likelihood scheme is an optimal deconvolution scheme while decision feedback and least square deconvolution schemes are suboptimal Our main concern throughout is to understand various theoretical and computational issues involved in developing DR scheme based on these methods

Salient features of the DR schemes based on the above-mentioned deconvolution methods are briefly described in the following paragraphs

1 Maximum Likelihood Deconvolution Scheme

We first derive a maximum likelihood (ML) recursive deconvolution algorithm (ML-RDA) based on Mitten s lemma which provides conditions for optimal decomposition of multistage decision processes This algorithm performs essentially the job of ML sequence estimation based on Viterbi algorithm However the derivation of the algorithm does not require the use of state representation and explicit invocation of Bellman s principle of optimality which are essential in deriving Viterbi algorithm The approach taken provides better insight into the ML deconvolution problem compared to the conventional VA application We have shown that if the dat v lues are assumed to

be real variables the recursive algorithm reduces to the familiar Gaussian elimin tion algorithm for solving a set of linear equations We can therefore view ML deconvolution algorithm as a generalized Gaussian elimination algorithm for minimizing banded Toeplitz quadratic objective functions

In the case of binary data we show that the ML RDA reduces to a special case of the algorithm given by Hammer et al for minimizing pseudo Bool an functions We further show that in the binary case the problem of reducing storage and computational complexity of ML RDA involves determination of minimal sum of products form of Boolean functions of several variables Therefore fast algorithm available for such purpose can be used to speed up computations and to reduce storage requirements Finally we show that the recursive algorithm derived for minimizing pseudo Boolean functions can also be used for ML-decoding of binary block codes Usefulness of the method is illustrated with the help of examples However much remains to be done in this direction

2 Decision Feedback Deconvolution Schemes

We next consider the possibility of incorporating decision feedback (DF) in DR schemes for BDT with the objective of achieving good performance without going in for ML-deconvolution which may not be economical in some practical situations

DF-DR schemes consist of a forward filter for minimizing the effect of future symbols a nonlinear decision device and a backward filter for eliminating the effect of past symbols This scheme may be viewed as Gaussian elimination method of solving linear equations using forward elimination and backward substitution with a nonlinear decision device interposed between forward elimination and back ubstitution tages

For deriving DF-DR sch mes we first minimize the banded Toeplitz quadratic function assuming that the data values are real This leads to the solution of banded Toeplitz system of linear equations We then derive three DF-DR chemes for BDT The first scheme is based on Cholesky decomposition of positive definite matrices This scheme however does not take noise v riance into account The second DF-DR scheme takes noise variance into account and is based on minimum mean square error criterion The third DF-DR scheme is based on minimum property of Fourier expansions over finite dimensional Hilbert spaces This scheme is useful when the channel autocorrelation matrix is illconditioned and when no knowledge of the noise variance is available

Finally we derive a DF-DR scheme by modifying Austin s detection theory approach We show that the modified Austin s method can also be viewed as Gaussian elimination method with a nonlinear decision device

Least Squares Deconvolution Schemes-Direct App oach

We then undertake a ritical study o l ast uares (LS) deconvolution using discrete Fourier transform (DFT) techniques The main f ature of our study is in providing a general treatm nt to the LS-deconvolution probl m using the notion of square completions of the channel onvolution ma rix We show that both LS solution and the con ntional DFT approach are particular cases of square completions of the chann l convolution matrix Moreover DFT approach to the d convolution probl m is a particular case of k-Circulant completion of the channel convolution matrix Based on k-Circulant completions we develop two data recovery schemes which are easily implementable and have redu ed computational complexity

Although DFT based deconvolution is widely used conditions under which such techniques provide optimal solution to deconvolution problem according to some criterion do not seem to have been explicitly m ntioned We investigate conditions on k Circulant completions which provide LS solution to the deconvolution problem Using the framework of quare completion of matrices we obtain conditions on square completion that provide Moore-Penrose inverse of a full rank matrix Such comple tions are called orthogonal completions Then it is shown that orthogonal k-Circulan compl tions are orthogonal k-Circulant matrices Using this result we show that for all practic l

ases of interest LS solution to the deconvolution problem can not be obtained via direct inversion of k Circul nt completions

4 Least Squares Deconvolution Scheme - Iterative Approach

Finally we con ider an ite ative d convolution scheme using Steepest Descent Algorithm (SDA) One of the purposes of this study is to examine the observ tion of several authors working with fixed step SDA in the DFT domain that the DFT domain iteration scheme converges faster than the sample domain scheme For this purpose we deriv SDA in the DFT domain from three dif rent view points We show that the DFT domain scheme does not alter the conv rgence rate it only helps in reducing computational complexity at each iteration However it is shown that k-Circulant comple tions can be used to obtain good estimates of the convergence parameter in the case of some examples These estimated values can be used either in the sample domain or in the DFT domain to accelerate convergence

Since the rror performance of each data recovery scheme strongly depends on the particular channel used we give an account of good channels for the purpose of comparing the performance of various DR sch mes Two criteria for good channels are considered The first criterion provide channels having maximum minimum distanc (MMD) called MMD-channels These channels re used as benchmarks for comparing the performance of various DR scheme used on actual channels

Determination of MMD-channels however is a difficult problem Therefore we consider the second class of good channels called maximum distance channels (MD-channels) which maximize the total distance A procedure is given for obtaining MMD-channels via MD-channel characterization For the purpose of comparing actual channels with MMD-channels we also develop a recursive algorithm for computing the minimum distance of a given channel It is shown that some of the MMD and MD channels are partial response channels

The performance of some of the data recovery schemes discussed earlier has been evaluated using computer simulation Performance curves showing the variation of probability of error with signal-to-noise ratio are given for a wide range of block lengths and typical channel examples

The simulation study brings out the following merits of BDT over continuous serial data transmission (CSDT)

1) If decision feedback idea is used to improve the performance of BDT systems error propagation is limited to atmost a block length in the worst case whereas in the case of DF-DR schemes for CSDT error propagation can extend over much longer duration Since in the range of medium signal-to-noise ratios the error propagation is a serious problem with DF DR schemes for CSDT systems BDT with DF DR schemes are to be preferred in such applications

2) With an appropriate choice of the block length BDT can be used to provide better trade-off between performance processing complexity and data rate

A drawback of BDT is that the effective data rate over the channel is less compared to CSDT for a given source rate The loss in data rate may be compensated by transmitting data over the channel at a higher rate This result in increased intersymbol interference among the block samples Our simulation studies however indicate that if the block length is properly chosen the degradation in performance of DR schemes for BDT with increased intersymbol interference will not be significant

CHAPTER 1

INTRODUCTION

In this thesis we study several data recovery (DR) schemes for block data transmission over linear time dispersive channels in the presence of noise. In block data transmission (BDT) the input data sequence is partitioned into blocks which are separated by guard zones of appropriate duration such that the channel output blocks do not overlap. Since the channel is assumed to be linear its outputs can be represented as convolution of the input data values and the channel impulse response. The operation of recovering data from the nonoverlapping blocks at the channel output is then essentially that of finite deconvolution in the presence of noise, that is, given the channel impulse response and the nonoverlapping channel output blocks which have been corrupted by noise, the problem is to recover the data at the receiver end in some optimum manner.

We formulate the deconvolution problem as minimization of a banded Toeplitz quadratic function with data as unknown variables. When the unknown variables are real, minimization of the quadratic function is straightforward. However, when the variables take values from a finite alphabet, as in the case of data transmission, it is difficult to provide computationally attractive solutions. The solution to the deconvolution problem is

optimal or suboptimal depending on the assumptions made on the unknown variables and the trategy aopted for solving the problem

We study maximum likelihood d cision f edback and least squares deconvolution methods n detail Among these schemes maximum likelihood (ML) scheme is an optimal deconvolution scheme while decision feedback (DF) and least squares (LS) deconvolution schemes a suboptimal Our main concern throughout is to understand va ious theoretical and computational issues involved in developing DR schemes based on these methods

1 1 *MOTIVATION*

Design and evaluation of DR sch mes for long strings of data transmitted over time dispersiv chann ls in the presence of noise is an importan topic of study in the area of digital communication Development of v rious data transmission and recovery cheme which date back to the pion ering work of Nyqui t [1] and which continue to draw a great deal of attention even today has been primarily motivated by a de ire to increase the d ta rate Thi is refl cted in the design of most of the available data transmission and recovery schemes In this connection continuous serial data transmission (CSDT) happens to be the mo t widely used data transmission scheme in which a continuous string of serial d a is transmitted over the channel

In the case of data transmission over time dispersive channels due to intersymbol interference (ISI) the signal samples at the receiver input become highly correlated. Therefore implementation of optimal DR schemes for CSDT systems becomes impractical as they require very large observation intervals which means prohibitively large delays and processing requirements. One method of circumventing these problems which has received considerable attention is to make decisions based on truncated observation intervals. This strategy reduces delay and processing time at the expense of increased probability of error. The trade-off however appears to be one sided from the point of view of the following three major demands any good data transmission system is expected to satisfy.

D1 Minimum probability of error. The probability of a detected data symbol being in error must be minimum.

D2 Minimum processing time. Computational complexity of the data recovery scheme must be minimum and the scheme must be easily implementable.

D3 Maximum data rate. It must be possible to transmit data values one after another with minimum time gap between adjacent symbols.

The foregoing discussion highlights the fact that there is still a need for investigating alternate ways of data transmission which will help in striking a better compromise between D1, D2 and D3.

An alternate method of reducing delay and processing requirement without sacrifying error performance is to reduce the observation interval by appropriately changing the data transmission format This may be achieved by using BDT format studied by Clark [2] - [4] We study various data recovery schemes for BDT with the following objectives in mind

1 A better understanding of the possible trade-off between performance processing complexity and data transmission rate in the context of BDT
2 A better appreciation of the merits and demerits of the existing DR schemes for CSDT
3 To critically analyse the advantages of BDT in situations where probability of error performance of the transmission system is more important than achieving maximum data rate at the cost of error performance as in the case of CSDT

1 2 REVIEW OF LITERATURE

To place our study in a proper perspective we shall first review the existing literature on DR schemes for CSDT and BDT Our interest is not to give a detailed account of the historical development of various DR schemes Instead we wish to highlight some of the key developments with the help of relevant literature It may be pointed at the outset that most of the work is in connection with CSDT references on BDT are very few We review the literature on CSDT first and consider BDT later

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1 2 1 Data Recovery Schemes fo Continuous Serial Data Transmission

The literature on DR schemes for CSDT in t e presence of ISI and noise is ri h with a wid variety of novel pproach s and related theoretical developments A number of well written books are available on this topic i xts by Bennet and Davey [5] Lucky Salz and W ldon [6] and Proakis [7] provide excellent coverage of most of the important topics in the area of CSDT longwith long lists of references the text book by Clark [4] provides wide coverage to both CSDT and BDT The collection of papers edited by Franks [8] on data communication provides at a place tw nty four important works on CSDT alongwith informative and well written long ditorial comments Important dev lopments that took place in the do i in of CSDT during 1968 73 are discussed with extensive bibliography in a survey article by Lucky [9] Most o the r port d data r covery schemes for CSDT are either some variant of one of th following thre schemes or their combinations

1 Linear Data Recovery Schem s

Th first and the most simple one is th linear DR scheme consi ting of a ma ch d filter mat hed to the channel followed by a transve sal filter whose output is ampl d and used in making decisions First s st matic study in this connection app ars to b du to Tufts [10] [11] Extending th cla sical

work of Nyquist [1] Tufts [10] has derived an optimal linear receiver structure using minimum mean square error (MSE) criterion As an extension of this work Smith [12] has obtained explicit solution to the optimum receiver when the transmitted waveform is subjected to an average power constraint The criterion of optimization however remains same as before

Deviating considerably from the previous approaches and using detection theory point of view Austin [13] has shown that linear receiver can be derived on the assumption that the data values are independent Gaussian random variables The resulting transversal filter in general is of infinite length However for practical purposes only finite length filters are employed and the tap coefficients are optimized by minimizing the MSE between the filter output at the sampling instant and the actual data value Due to their simplicity finite transversal filters have been used extensively for the purpose of automatic channel equalization [14]-[22] as well as for adaptive equalization [23]-[28]

2 Decision Feedback Data Recovery Schemes

The second DR scheme known as decision feedback equalizer (DFE) [13] [29] consists of a matched filter sampler forward filter backward filter and a nonlinear decision device DFE makes use of past decisions to improve performance However as the actual decisions are always prone to error (except possibly

at hıgh sıgnal-to-noıse ratıo (SNR)) DFE suffers from the problem of error propagatıon DFE structure can be derıved on the assumptıon that the past symbols are error free and that the future symbols are ındependent Gaussıan random varıables Using thıs assumptıon Austın [13] has derıved he structure of the DFE The resultıng forward fılter ıs ın general of ınfınıte length However a ın the case o lınear DR schemes ıt ıs common practıce to use only fınıte length and the tap co ffıcıents are optımız d usıng MSE crıterıon DFE has sınce been studıed from arıous vıewpoınts [30]-[37]

3 Maxımum Lıkelıhood Data Recovery Scheme

The thırd DR scheme known as maxımum lıkelıhood DR scheme ıs due to Forney [38] He employed Vıterbı algorıthm (VA) [39]-[42] to allevıate the computatıonal complexıty assocıated wıth ML sequence estımatıon of data Th ML sequence stımator mınımı es the probabılıty of sequenc error when all the s quences are qually lıkely We wısh to note here that an algorıthm sımılar to VA has been dev loped by Kobayashı [43] for ML decodıng of a partıcular type of PAM sıgnal before the publıcatıon of the Forn y s now classıcal work [38]

For ML s quence estıma ıor the VA requıres ntıre observatıon values For long dat s reams such a sch m therefore becomes un tractıve bo h from the poınt of vıew of computatıonal complexıty and storage r quırements To make the VA

useful dccisions ar mad after a fixed numb r of observations becom s av ilaol this nuno r is fixed empirically on th basis of channel memory with the ssumption that d gradation duc o such premature dccision is n glig ble [30] However since the pcrform nce degradation strongl d pends on the channel and noise such an assumption n cds c reful c amin tion

Aft r Forno/ s work all for s on ML DR schemes seem to have been dir cced towards further reducing the complexity of VA [4]-[52] Specific lly since the computational complexity of the VA grow xponenti lly with the channel memory efforts hav been nad to shorten he channel impulse response duration by using prefilters

1 2 2 Data Pecovery chem s for Block D a Transmission

Clark [2] [4] has inves ig t d a number of adaptive detection proces for BDT syst n In his work the BDT has been referred to as transmission o o thogonal group of signal elen nts by v rtue of h fact that at the channel output such groups do not overlap The rategy used by him is based on the recognition th t the dct ction of l ments in a group can be carried out by solving a set or l near simultaneous equations using iterative techniques In particular the adaptive detection processe studied by him ial o u e of the point Gauss-Seidel iteration n thoa He ha also considered several modifications of thi basic detection scheme to take care of the

dı crete nature of t͡ıe ɑata From the extensıve sımulatıon worl carrı°d out by hım and hıs group [4] ıt app ars tl at such ıodıfıc tıons lead to effıcıent and cc t effectıve DR schemes whose perfoımance ıs comp raɒle to L approach Moreover such modıfıcatıons do not requıre e cessıve number of sequentıal oɔeratıon Extendıng thı approa h Clark and Ghanı [53] have studıed detectıon processe for multı level sıgnallırg schemes Sımulatıon resul^s presenl d by them show that good toler nce o noıse can be achıeveɑ ɒy such detectıon proc sses whıch do not requıre exces ıve number of sequentıal opeıatıons

Befoıc wıncıng up thıs sectıon we wısh to take note of th work o Kleın anc Iolf [54] ın whıch BDT ıs consıdeıed from a dıfferent poınt of vı w They have ınvestıgated certaın as p cts of h possıbılıty of desıgnıng a receıver ıılter such that the overall response of the channel and the receıver fılter resembles a cyclıc codc gcnerator However thıs lıne of thought docs not seem to have bcen pursueɑ further

1 3 KEY R_SULTS AND OB_RVATIO

As mentıoned earlı the basıc approach ın our study s to vıew th d a recoveıy proble ı as a fınıte leconvolutıon proɒlem ın the pr sence of noıse alıent featur s of the study are brıefly descrıɒed ın th following paragraphs

1 Maximum Likelihood Deconvolution Scheme

We first derive a ML recursive deconvolution algorithm (ML-RDA) based on Mitten's lemma [55] which provides conditions for optimal decomposition of multistage decision processes. This algorithm performs essentially the job of ML sequence estimation based on VA. However, the derivation of the algorithm does not require the use of state representation and explicit invocation of Bellman's principle of optimality [56] which are essential in deriving VA. The approach taken provides better insight into the ML deconvolution problem compared to the conventional VA application. We have shown that if the data values are assumed to be real variables the recursive algorithm reduces to the familiar Gaussian elimination algorithm for solving a set of linear equations. We can therefore view ML-deconvolution algorithm as a generalized Gaussian elimination algorithm for minimizing banded Toeplitz quadratic objective functions.

In the case of binary data we show that the ML-RDA reduces to a special case of the algorithm given by Hammer et al. [57] for minimizing pseudo Boolean function. We further show that in the binary case the problem of reducing storage and computational complexity of ML-RDA involves determination of minimal sum of products form of Boolean functions of several variables. Therefore fast algorithms available for such purpose can be

used to speed up computations and to reduce storage requirements Finally we show that the recursive algorithm derived for minimizing pseudo Boolean functions can also be used for ML-decoding of binary block codes Usefulness of the method is illustrated with the help of examples However much remains to be done in this direction

2 Decision Feedback Deconvolution Schemes

We next consider the possibility of incorporating decision feedback in DP schemes for BDT with the objective of achieving good performance without going in for ML-deconvolution which may not be economical in some practical situation

DF-DR schemes consist of a forward filter for minimizing the effect of future symbols a nonlinear decision device and a backward filter for eliminating the effect of past symbols This scheme may be viewed as Gaussian elimination method of solving linear equations using forward elimination and backward substitution with a nonlinear decision device interposed between forward elimination and back substitution stages

For deriving DF-DR schemes we first minimize the banded Toeplitz quadratic function assuming that the data values are real This leads to the solution of banded Toeplitz system of linear equations We then derive three DF-DR schemes for BDT The first scheme is based on Cholesky decomposition of positive definite matrices This scheme however does not take noise

variance into account The second DF DF scheme takes noise variance into account and i based on minimum mean square error criterion The third DF-DR scheme is based on minimum property of Fourier expansions over finite dimensional Hilbert spaces [58] This scheme is useful when the channel autocorrelation matrix is illconditioned and when no knowledge of the noise variance is available

Finally we derive a DF-DP scheme by modifying Austin s d ection theory approach [1] We show that the modified Austin s method can also be viewed as Gaussian elimination method with a nonlinea decision device

3 Least Squares Deconvolution Schemes-Direct Approach

We then undertake a critical study of least squares deconvolution using discrete Fourier transform (DFT) techniques The main feature of our study is in providing a general treatment to the LS-deconvolution problem using the notion of square completions of the channel convolution matrix We show that both LS solution and the conventional DFT approach are particular cases of squai completions of the channel convolution matrix Moreover DFT approach to the deconvolution problem is a particular cas of k-Circulan completion o the channel convolut on matrix Based on k-Circulant completions we develop two data recov ry schemes which are easily implmentable and have reduced computational complexity

Although DFT based deconvolution is widely used [59]-[60] conditions under which such techniques provide optimal solution to deconvolution problem according to some criterion do not seem to have been explicitly mentioned We investigate conditions on k-Circulant completions which provide LS solution to the deconvolution problem Using the framework of square completion of matrices we obtain conditions on square completions that provide Moore-Penrose inverse of a full rank matrix Such completions are called orthogonal completions Then it is shown that orthogonal 1-Circulant completions are orthogonal 1-Circulant matrices Using this result we show that for all practical cases of interest LS solution to the deconvolution problem can not be obtained via direct inversion of 1-Circulant completion

4 Least Squares Deconvolution Scheme-Iterative Approach

Finally we consider an iterative deconvolution scheme using steepest descent algorithm (SDA) One of the purposes of this study is to examine the observation of several authors [21] [22] [64] [66] working with fixed step SDA in the DFT domain that the DFT domain iteration scheme converges faster than the sample domain scheme For this purpose we derive SDA in the DFT domain from three different viewpoints We show that the DFT domain scheme does not alter the convergence rate it only helps in reducing computational complexity at each

iteration However it is shom that k-Circulant completions can be used to obtain good stimates of the convergen e parameter in the case of some examples Th se estimated values can be us d ither in the sample domain or in the DFT domain to acc lerate convergence

Since the error p rformance of each DI schem trongly depends on the particular channel used we give an account of good channels for the purpos of comparing the performance of v rious DR scheme Two criteria for good channels are con- idered Th first cri erion provide channel having maximum minimum distance (MMD) c lled MMD channels Th se channels are u ed a be chma ks for comparing the performance of various DR schemes used on ac al chann ls

D termination of MMD-chann ls however is a difficult problem Therefor w consid r the second class of good channels called maximum distance channels (D channel) which maximiz the total distance A procedure is given for obtaining MD-channels via MD-channel characterization For the purpo of comparing actual channels with MMD-channels we also d velop a recursiv algo ithm for computing th minimum dis- t nce of a given chann l It is shown th t some o the MMD and MD channels are partial re ponse channels

The performanc of some of he data recovery schemes discussed arlier has been evaluated using compu er simulation

Performance curves showing the variation of probability of error with signal-to-noise ratio are given for a wide range of block lengths and typical channel examples

The simulation study brings out the following merits of BDT over CSDT

1) If decision feedback idea is used to improve the performance of BDT systems error propagation is limited to almost a block length in the worst case whereas in the case of DF-DR schemes for CSDT error propagation can extend over much longer duration Since in the range of medium signal-to-noise ratios the error propagation is a serious problem with DF-DR schemes for CSDT system BDT with DF DR schemes are to be preferred in such applications

2) With an appropriate choice of the block length BDT can be used to provide better trade-off between performance processing complexity and data rate

A drawback of BDT is that the effective data rate over the channel is less compared to CSDT for a given source rate The loss in data rate may be compensated by transmitting data over the channel at a higher rate This results in increased ISI among the block samples Our simulation studies however indicate that if the block length is properly chosen the degradation in performance of DR scheme for BDT with increased ISI will not be significant

1 4 OPGANIZATION OF THE THESIS

Be ides this chapter the thesis consists of 8 chapters Chapter 2 gives a detailed account of the assumption made in the study chann l model considered a discussion of the BDT format and problem formulation In Chapter 3 we develop a re cursive algorithm for max mum likelihood deconvolution of data and study the application of this methodology for decoding or error correcting binary block codes Decision f edback DR schemes are studied in Chapter 4 Application of k-Circulant matrices for designing cost effectiv DR schemes is studied in Chapter 5 In Chapter 6 we discuss conditions on k-Ci culants for providing least squares solution to the data recovery pro blem Application of steepest descent algorithm for LS-deconvolution is dis ussed in Chapter 7 In Chapter 8 we study the performance of some of the DR schemes developed in the thesis Chapter 9 gives conclusions and suggestions for further research

1 5 NOTATIONS AND CONVENTIONS

In this sec ion we wish to mention some of the notations and conventions which are common to all the chapters of the thesis Rest of he notations and conventions are explained in a list included after the li t of figures Lower case letters with a bar below are used to denote sample domain or time do-main vec ors For example $\underline{a}$ $\underline{h}$ $\underline{y}$ are sample domain vectors

Upper case letters with a bar below r present DFT or frequency domain vctors of the corresponding samole or time domain v ctors Upper cas lett rs without a bar below except R and C rep esent matrices I_m and O_{mxn} respec ively denote identity matrix of order m and null matrix of order mxn R^m and C^m denote m-dimensional real and complex vector spaces respectively We use R(A) to denote the range spa e of A - the space spanned by the columns of A and N(A) to denote the null space of A- the space of ll vectors $\underline{x}$ such that $A\underline{x} = \underline{o}$ Lower case letters without under bar are use to denote variable nd constants Transpose of a matrix B is denoted as B^T If B is a complex matrix B denote complex conjugate transpose of B Similarly $\underline{B}^*$ d notes complex conjugate transpose of vector $\underline{B}$

$|| \underline{x} ||$ denotes the Euclidean norm of the vector $\underline{x}$ given by

$$|| \underline{x} || = (\underline{x}^* \underline{x})^{1/2}$$

$|| A ||$ denotes norm of matrix A When A is symmetric $|| A || = \max_{1<i<n} | \lambda_i |$ where $|\lambda_i|$ is the absolute value of the i h eigenvalue of n n n matrix A

End o h proof of theorems is indicated by double star (**) Computational complexity of algorithm is denoted by O(g(n)) which is to b read as order of g(n)

CHAPTER 2

BLOCK DATA TRANSMISSION

In this chapter we set the stage for the development of later chapters. The assumptions made in the study are given in Section 2.1. Discrete model of the channel used in the study is discussed in Section 2.2. Block data transmission format and some of its details are given in Section 2.3. Finally in Section 2.4 we show that the deconvolution problem is equivalent to minimization of positive definite Toeplitz quadratic function with data as unknowns.

2.1 ASSUMPTIONS

Consider the data transmission system shown in Figure 2.1. In the figure input data stream to the transmitter filter is a sequence of impulses $\{x_i \delta(t - iT_s)\}$ where T_s is the intersymbol duration and $\delta(t)$ denotes an unit impulse function. x_i denotes the value of the ith transmitted symbol. The transmission path is a linear baseband channel consisting of a modulator, bandpass channel and a demodulator. Noise is added at the output of the channel.

For the purpose of study we make following assumptions regarding the data transmission system.

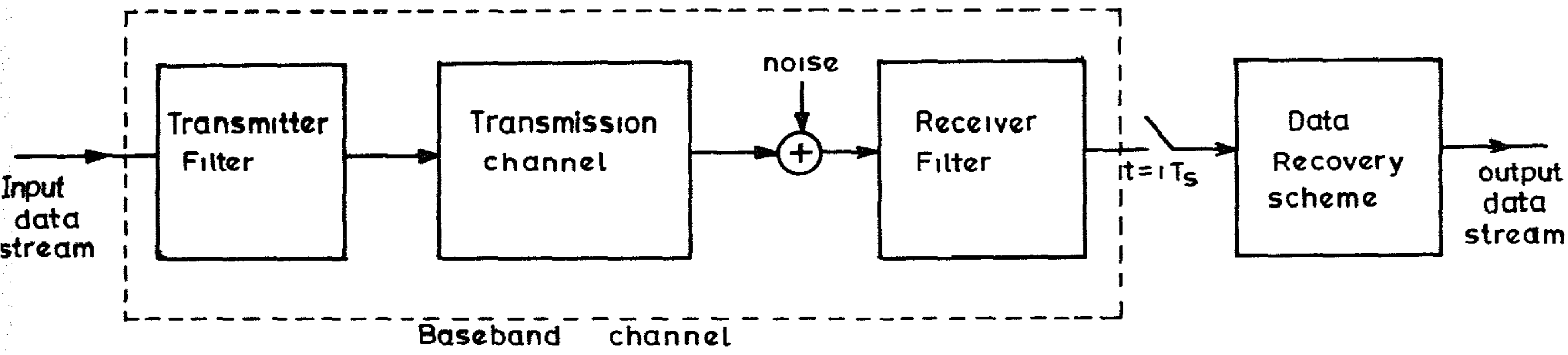


FIG 2 1 Block diagram of data transmission system

1 A synchronous data transmission system is assumed That is the receiver has knowledge of the correct mpling in tants

2 The ba eband channel is linear and i is characteriz d by ar impulse response $h(t)$ In nost o th prac ical cases channels can be modelled either as time invariant or as lowly ime varying In the case of tin varying channel data recov ry sch mes must be made adaptive V ho ever will not s udy this aspect in this the s

3 The channel impuls response is of fir e duration This assumption is met by mo t o the practic l ch nnels Al hough for a bandlimited channel the impulse response is of infinite duration in mo t of the case mpulse response becomes n gli-c ble after a finite duration

4 Data symbols are statistically independent and take on valu with equal probability from the set $\{\pm 1\ \pm 3\ \pm 5\ \ \ \pm(q-1)\}$ where q is an even integer This assumption is rarely met by any practical com unication syst m However it can be r alized t some extent with the help of scramblers at the tran mitter end The effect of scrambling can be removed by descrambling the data symbols at the receiver end (see Clark [4] page 178)

5 Noise samples are statistically independent and each sample is a Gaussian random variabl with zero mean and variance $N_0/$ Th white Gaussian noise assumption is widely used as it considerably simplifies mathematical analysis and provides worst case performance

It ma/ be noted that although some of these assumptions may not b valid in prac ic th / provide a us ful f amework or formulating and analysing the problem

2 2 DISCPETE TIME MODEL FOP ISI CHANNELS

Due to the pr sence of th sampler aft r the receiving filter (see Figure 2 1) it is conv nient to anal/se the base band data transmission s/stem with a discrete time equivalent model When a finit number of da a ymbols are transmitted over finite ISI channels in he presence o whit G ussian noise it can be shown [38] [7] that he channel mod l shown in Figure 2 2 provides an quivalent model for the estimation of th input sequence withou any loss o rror performance For this to happen transfer function of the transversal filter I(z) must be chosen such that

$$X(z) = H(z)\ H^*(z) \qquad (2\ 1)$$

where X(z) is the z-transform of the autocorrelation function of the channel impulse response h(t) and H(z) is a polynomial having roots of X(z) which are in ide the unit circle

Clark [3] has consider d chal rel responses deriv d in a d fferent manner H as the that effective multipath distor tion introduced by the transmission path can be modelled as an quival nt sampled impulse response of the linear transmission path The ov rall sampled r spons of th channel is obtained

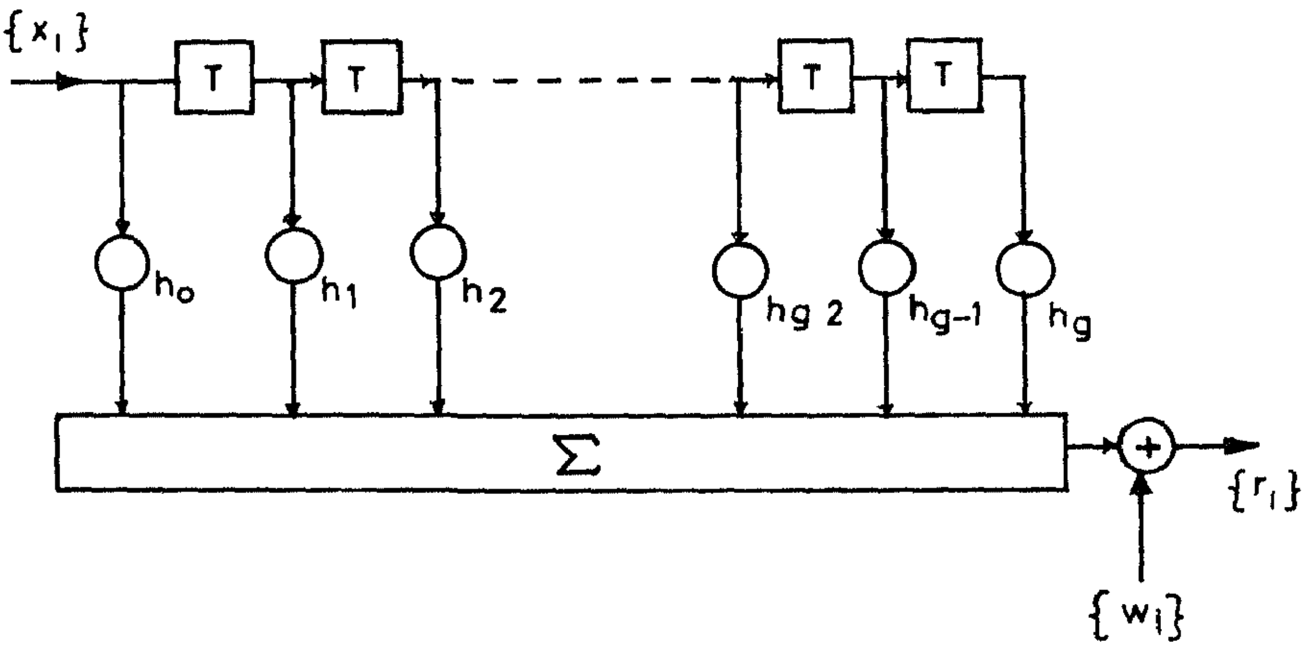


FIG 2 2 Discrete model of finite impulse response channel

by considering identical transfer functions for bo h transmitter and receiver filters

For the purpose of study we use the discrete time model for finite ISI channels sho m in Figure 2 The model consists of a finit length transver l ilt r who e tap coefficients reflect the ffec o transmitter filter cha mel and receiving filter Flexibility in the choi e of the tap coefficients to take care of a w d class of channels including channel variations (by making the tap coefficients vary with time) nd the imple manner in which it captures the es en ial idea of ISI and noise are the major onsiderations for u ing t e model in the s udy

In Figure 2 2 $\{x_i\}$ denotes a s quence of statistically independent data s/n ols $\{w_i\}$ denotes a sequence of independent Gau sian random variables with mean zero and variance $N_0/2$ h_i $i = 0$ 1 g are the ch nnel param t s indicates the number of ISI components and it depends upon h data rate and duration o the channel impul e response As the data rate increases g also increases

From the chann l mod l f Figur 2 2 it is easy to see that th output sequenc $\{r_i\}$ has the following r presentation

$$r_i = \sum_{k=0}^{g} h_k x_{i-k} \quad w_i \qquad i = 0\ 1\ 2 \qquad (2\ 2)$$

Data recov r/ problem in h case of CSDT systems is essentially that of solving the et of equation giv n by (2 2) for x_1 s In the ca e of long strings of data this becomes a di ficult problem due to the large number of equations which are to be solved n h pr sence of noise Block data transmission format which overcomes this problem is described in the following ection

2 3 BLOCK DAT/ TRANS ISSION FORMAT

In BDT blocks of data of l ngth m wi h each block followed by g z ros is transmitted through the ch nnel If the source data rate s $1/T_s$ symbols p r second due to the g zero following each block th ffective rate of transmis ion / ll be $m/(nT_s)$ symbols per second where $n = m+g$ On the other hand no reduction in data rate is xperienced if he data symbols are transmitted through th channel at a rate of $1/T_c$ symbols per second where T_c is obtained a L t g b chosen such that the channel dispe sion duration (T_h) does not e ceed $(g+1)T_c$ seconds That is

$$(g+1)T_c > T_h \qquad (2\ 3)$$

Then T_c i giv r by

$$T_c = n\, T_s/(m+g) \qquad (2\ 4)$$

ubs ituting the v lue of T_c from (2 4) n (2 3) we obtain

$$g > m\,(T_h - T_s)/(mT_s - T_h)\ , \quad mT_s > T_h \qquad (2\ 5)$$

Due to the presence of g zeros following each transmitted block the channel is forced into zero state before the arrival of a fresh block at the input Thus the effect of the channel is confined to blocks of length n where n = m+g Therefore ISI does not propagate from one block to the subsequent ones Due to this feature blocks can be processed independently at the receiver to recover data value Note that two buffers will be needed of which one will be receiving the next block while the other holds the current block for processing

Figure 2 3 helps to make some of the ideas expressed in the previous paragraphs more clear In Figure 2 3a a rather simplistic channel is shown for the purpose of illustration The data duration is assumed to be one third of the duration of the impulse response Output of the channel shown in Figure 2 3a to an arbitrarily chosen string of binary (±1) data is shown in Figure 2 3b Figure 2 3c shows the corresponding overlapping pulse responses Figures 2 3d and 2 3e show the corresponding situation in the case of BDT with increased data rate over the channel In this case m = 6 g = and n = 10 Moreover we obtain $T_c = 0\ 6\ T_s$ as the data transmission rate over the channel From the Figures 2 3d and 2 3e it can be seen that the blocks get separated at the channel output

Certain features of the inequality (2 5) can be made clear with a graphical representation For this purpose

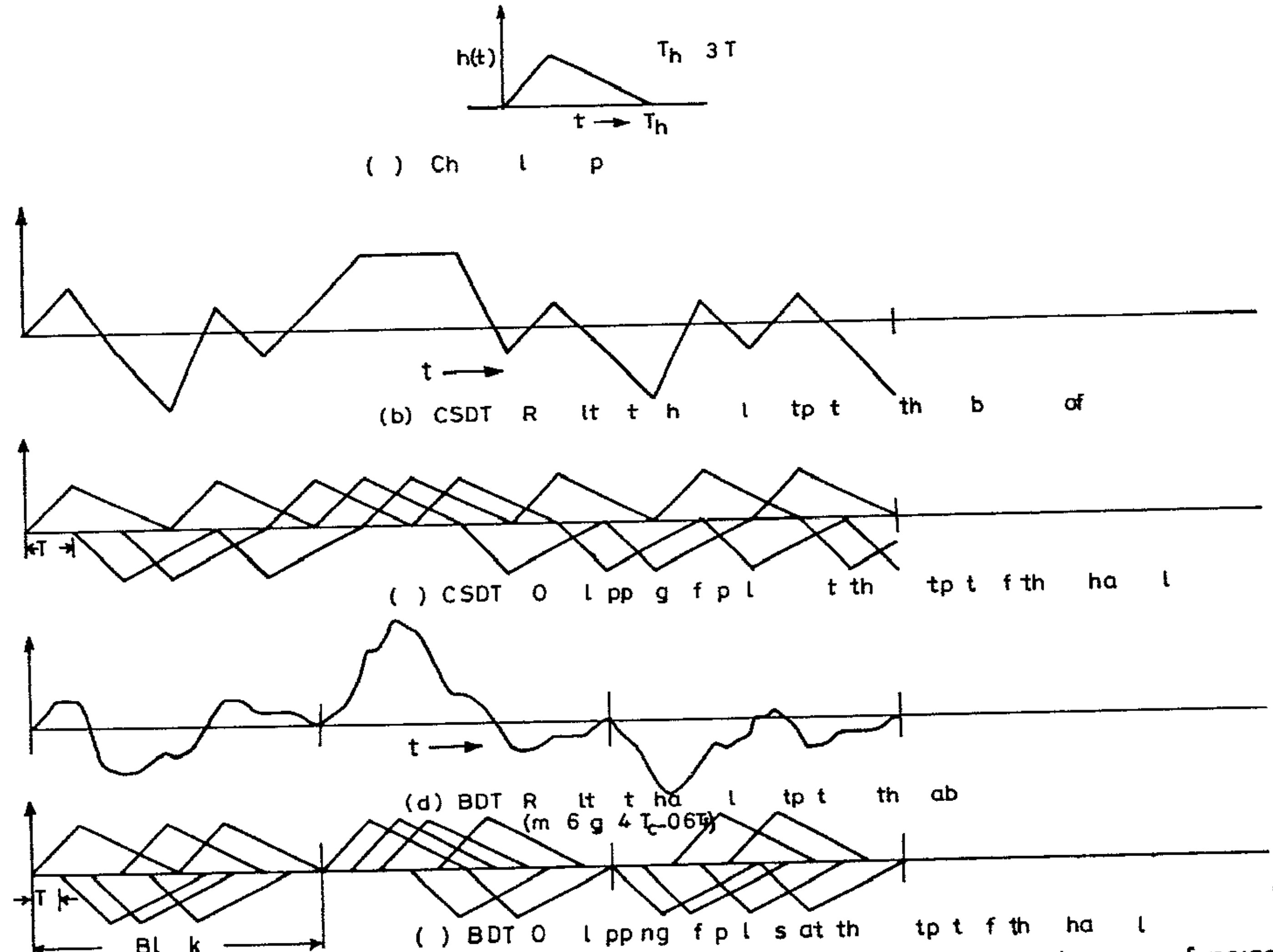


FIG 2 3 Response of a channel to CSDT and BDT in the absence of noise

let $p = T_h/T_s$ Then (2 5) becomes

$$g > (p-1)\, m/(m-p) \qquad m > p \tag{2 6}$$

In Figure 2 4 g i plotted against m for various values of p when g is given by

$$g = (p-1)\, m/(m-p) \qquad m > p$$

Note that although m and g are integers in Figure 2 4 in order to avoid too many symbols g is plotted as a continuous function of m for m > p From the figure it is clear that for a given p m must be sufficiently large for g to be small Under this condition T_c will be close to T_s We also observe that the required value of g levels off fast after dropping rapidly in the beginning This means that the effect of ISI among the members of a block will not become too large if moderate value of m is chosen

2 4 THE DECONVOLUTION PROBLEM

In this section we formulate the deconvolution problem Let h be an n-length vector with first g elements as the channel parameters and rest of the elements zero Let x denote an m-length data vector and r an n length observation vector In terms of these vectors the convolution relationship given by (2 2) can be written as

$$\underline{r} = H\,\underline{x} + \underline{w} \tag{2 7}$$

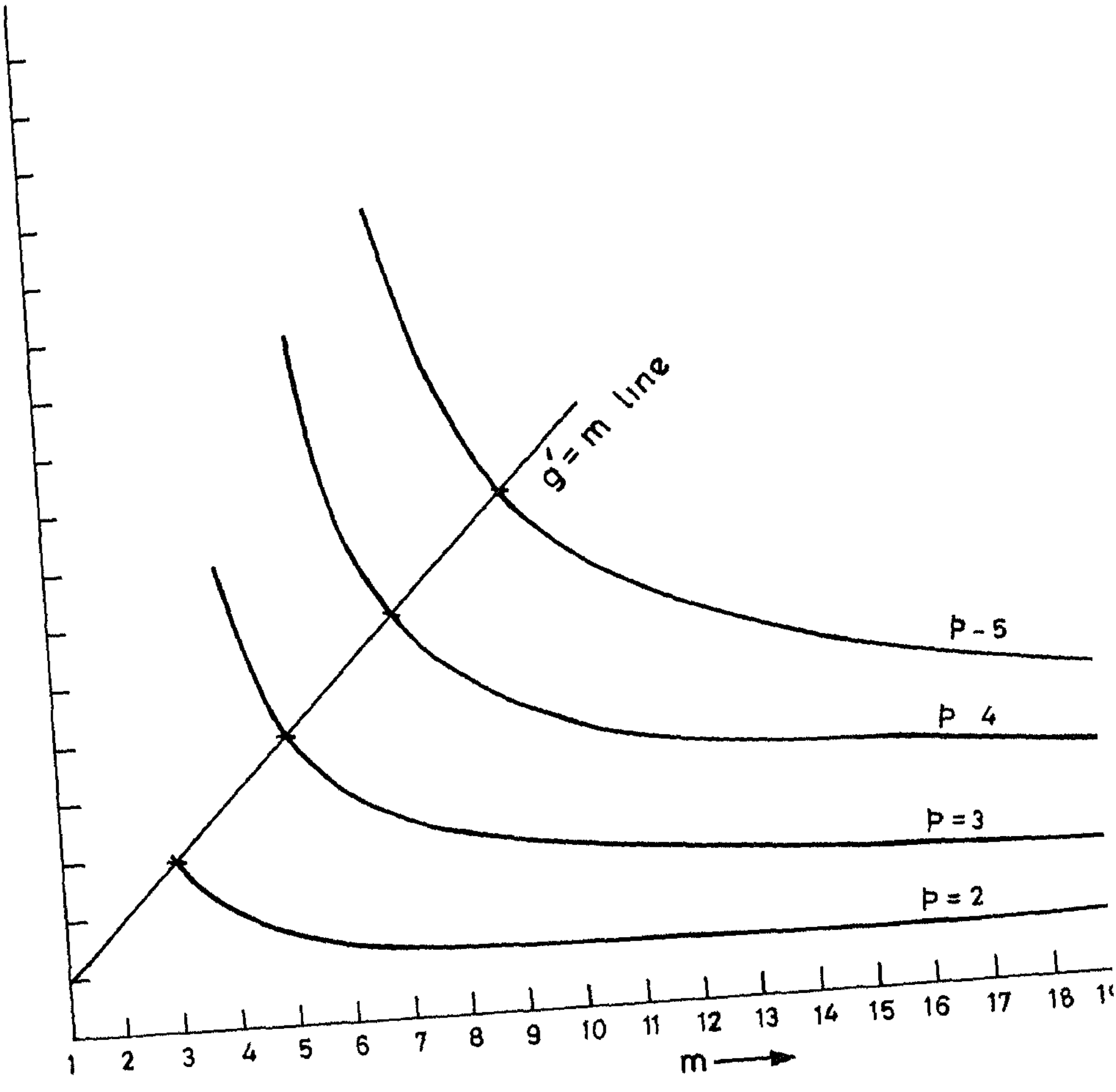


FIG 2 4 Plot of g' versus m

where H is the convolution matrix with h as the first column. For the same illustration H matrix is shown below

$$
H = \begin{bmatrix}
h_o & 0 & 0 & & 0 \\
h_1 & h_o & 0 & & \\
 & h_1 & h_o & & \\
 & & h_1 & & 0 \\
h_g & & & & h_o \\
0 & h_g & & & \\
 & 0 & h_g & & \\
 & & & & \\
0 & 0 & 0 & & h_g
\end{bmatrix}
$$

The (i j)th element of H is given by

$$
h_{ij} = \begin{cases} h_{|i-j|} & i \leq j \\ 0 & \text{for } i>j \text{ and } i-j > g \end{cases}
$$

$$
i = 1\ 2\ 3 \qquad n \qquad j = 1\ 2\ 3 \qquad m
$$

The problem of determining x given r and $\underline{h}$ or finding $\underline{h}$ given $\underline{r}$ and $\underline{x}$ is called finite d convolution problem in the presence of noise

No problem has received as much attention as the finite deconvolution problem due to its appearance in a wide variety of applications [59]-[61] [67]-[71] However most of the methods available for dealing with the problem assume that the unknown variables are real Whereas in the context of BDT the data vector x take discrete values from a finite alphabet Therefore in this case we have to deal with a combinatorial deconvolution problem in which the space of all possible solutions does not possess the nice properties of R^n to which real vectors belong One straightforward method of deconvolution is to search through all possible solution However even in the binary case searching through all possible 2^r vectors becomes out of hand for moderate values of m Moreover due to the presence of noise no exact solution is possible in general The usual approach taken under such situations is to choose a vector x which optimize some suitable performance criterion Depending on the constraints put on the solution space one obtains different solutions to the deconvolution problem in the presence of noise

In data communications the most sought after criterion is the minimization of the error probability For the case of equally likely data vector it can be shown that the maximum likelihood (ML) sequence estimation provides minimum probability of error [7] The ML estimate of x is that vector which maximizes the conditional density $p(y/x)$

Under the assumptions made in section 2 1 the conditional density function of r given a particular data vector x can be shown to be

$$p(r/x) = \prod_{i=1}^{n} \frac{1}{\sqrt{\pi N_o}} \exp\left(\frac{1}{N_o} (r_i - \sum_{l=o}^{g} h_k x_{i-k})^2\right)$$

it is easy to see that the above density function is maximized by minimizing the quadratic function

$$F(x) = (\underline{r}-H\underline{x})^T (\underline{r}-H\underline{x})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i x_j a_{|i-j|} - \sum_{i=1}^{m} b_i x_i + \sum_{i=1}^{m} r_i^2 \qquad (2\ 8)$$

where $a_{|i-j|}$ is the $(i\ j)$th element of H^TH and b_i is the ith element of the vector $2H^T\underline{r}$

Therefore the deconvolution problem in our case is equivalent to minimization of the quadratic function given by (2 8) subject to the condition that the unknown variables take values from a finite alphabet Since H is a convolution matrix it is easy to see that H^TH is a symmetric Toeplitz matrix We wish to note that a square matrix D is called Toeplitz if $d_{ij} = d_{|i-j|}$ where d_{ij} is its $(i\ j)$th element In addition as g is less than n H^TH is also a band matrix Thus the deconvolution problem is essentially that of minimizing a banded Toeplitz quadratic function subject to different constraints on the unknown variables depending on various performance criteria

CHAPTER 3

MAXIMUM LIKELIHOOD DECONVOLUTION

In this chapter we use Mitten s lemma [55] to derive a maximum likelihood recursive deconvolution algorithm (ML-RDA) which essentially performs the job of Viterbi algorithm (VA) [38] Mitten s lemma may be viewed as a more formal statement of Bellman s principle of optimality [56] and it clearly stipulates the conditions for optimal decomposition of multistage optimization problems For the sake of easy reference Mitten s lemma is given in Appendix A

The chapter is organized in the following manner In Section 3 1 we use Mitten lemma to develop a general recursive algorithm for minimizing decomposable nonlinear function of several variables A specific case when the nonlinear function is a quadratic function is taken up in Section 3 1 1 It is shown that minimization of a quadratic function of real variables using the recursive algorithm is equivalent to Gaussian elimination algorithm for solving a set of linear simultaneous equations The discrete version of the algorithm is discussed in Section 3 1 2 The ML-RDA for a banded Toeplitz quadratic function (BTQF) is given in Section 3 2 For the purpose of comparison in Section 3 3 we discuss VA

for mınımızıng BTQF Strategıes for allevıatıng storage and computatıonal requırements are dıscussed ın Sectıon 3 4 Applıcatıon of th se stratcgıes to Bınary case ıs consıdered ın detaıl ın Sectıon 3 4 1 Applıcatıon of the recursıve algorıthm for decodıng of block error correctıng code ıs consıdered ın Sectıon 3 5

3 1 RECURSIVE ALGORITHM FOR MINIMIZATION OF NONLINEAR FUNCTION OF SEVERAL V PIABLES

In order to ıllustrate th prıncıpl of ML RDA we fırst gıve a method for mınımızıng a nonlınear functıon of n varıables Let $F(x_1\ x_2 \qquad x_n)$ be a functıon to be mınımızed We assum that th functıon ıs decomposable ın the manner ındıcated below

$$F_1(\underline{x}) = F(x_1\ x_2 \qquad x_n) = U_1(x_1\ x_2 \qquad x_n) + V_1(x_2\ x_3 \qquad x_n) \qquad (3\ 1)$$

Note that V_1 ıs func ıon of $x_2\ x_3 \qquad x_n$ alone Now suppose that ınst ad of mınımızıng $F(\underline{x})$ wıth respect to $\underline{x}$ we can mınımıze U_1 wıth respect to x_1 fırst and then mınımıze the combıned functıon $\min\limits_{x_1} U_1(x_1\ x_2 \qquad x_n) + V_1(x_2\ x_3 \qquad x_n)$ wıth r spect to $x_2\ x_3 \qquad x_n$ Put dıfferently we wısh to compute P_2 where

$$P_2 = \min_{x_2 \qquad x_n} [V_1(x_2\ x_3 \qquad x_n) + \min_{x_1} U_1(x_1\ x_2 \qquad x_n)] \qquad (3\ 2)$$

At this stage a valid question to ask is whether or not $P_2 = \min_{x_1} \cdots \min_{x_n} F_1(x_1, x_2, \ldots, x_n)$? If the answer is yes then effectively we have decomposed the n variable minimization problem into a single variable minimization problem followed by an n-1 variable minimization problem. Thus it is essential to know whether we can distribute min operator as indicated in (3.2) without affecting the optimum solution.

In Lemma 3.1 we show that whenever $F_1(x_1, \ldots, x_n)$ is decomposable in the manner given by (3.1)

$$P_2 = \min_{x_1} \cdots \min_{x_n} F_1(x_1, x_2, \ldots, x_n)$$

which effectively proves that the min operator can be distributed in the manner shown in (3.2). We wish to note that Lemma 3.1 is a special case of the more general decomposition result given by Mitten [55] for multivariable optimization problems. A proof of Mittens result for two variables may be found in [56].

Lemma 3.1. Given (3.1) and (3.2) $P_2 = \min_{x_1} \cdots \min_{x_n} F_1(x_1, x_2, \ldots, x_n)$

Proof. Let $P_1 = \min_{x_1} \cdots \min_{x_n} F_1(x_1, x_2, \ldots, x_n)$

From the definition of minimum value we have

$$P_2 \geq P_1 \qquad (3.3)$$

Whenev r $U_1(x_1\ x_2 \qquad x_n) < U_1(x_1\ x_2 \qquad x_n)$ with $x_2\ x_3 \qquad x_n$ fix d i follows from (3 1) (sinc $V_1(x_2\ x_3 \qquad x_n)$ is fixed) that

$$F_1(x_1\ x_2 \qquad x_n) < F_1(x_1\ x_2 \qquad x_n) \qquad (3\ 4)$$

But fo each value of $x_2\ x_3 \qquad x_n$

$$U_1(x_1\ x_2 \qquad x_n) = \min_{x_1} U_1(x_1\ x_2 \qquad x_n) \leq U_1(x_1\ x_2 \qquad x_n)$$

whe x_1 is th valu of x_1 at which $U_1(x_1\ x_2 \qquad x_n)$ attains minimum value when $x_2\ x_3 \qquad x_n$ are fixed

Ther fore from (3 4) w c t

$$F_1(x_1\ x_2 \qquad x_n) \leq F_1(x_1\ x_2 \qquad x_n)$$

for all x_1 wi h $x_2\ x_3 \qquad x_n$ fixed (since $V_1(x_2\ x_3 \qquad x_n)$ is fixed)

Hence

$$F_1(x_1\ x_2\ x_3 \qquad x_n) \leq \min_{x_1} F_1(x_1\ x_2 \qquad x_n)$$

H nc $P_2 = \min\limits_{x_2 \quad x_n} [V_1(x_2\ x_3 \qquad x_n) + \min\limits_{x_1} U_1(x_1\ x_2 \qquad x_n)]$

$$= \min_{x_2 \quad x_n} F_1(x_1\ x_2 \qquad x_n) \leq$$

$$\min_{x_2 \quad x_n} \min_{x_1} F_1(x_1\ x_2 \qquad x_n) = P_1$$

therefor $P_2 \leq P_1$

From (3) we thus get $P_2 = P_1$

Let x_1^{**} be the value of x_1 which minimizes $U_1(x_1\ x_2 \quad x_n)$ for a fixed $x_2 \quad x_n$ No that x_1 b comes a function of $x_2\ x_3 \quad x_n$ Let

$$x_1 = g_1(x_2\ x_3 \quad x_n)$$

Substituting for x_1 in $F_1(\ _1\ x_2 \quad x_n)$ the minimization probl m now reduces to a n-1 v riable minimization problem

$$\min_{x_2 \quad x_n} F_2(x_2\ x_3 \quad x_n) \quad \text{where} \quad F_2(x_2\ x_3 \quad x_n) = F_1(x_1'\ x_2 \quad x_n)$$

Assuming again that F_2 is decomposable we can write F_2 as

$$F_2(x_2\ x_3 \quad x_n) = U_2(x_2\ x_3 \quad x_n) + V_2(x_3\ x_4 \quad x_n)$$

Using Lemma 3 1 we can write

$$\min_{x_2 \quad x_n} F_2(x_2\ x_3 \quad x_n) = \min_{x_3 \quad x_n} [V_2(x_3\ x_4 \quad x_n) + \min_{x_2} U_2(x_2\ x_3 \quad x_n)]$$

Let x_2 be the value of x_2 which minimizes $U_2(x_2\ x_3 \quad x_n)$ Then

$$x_2^* = g\ (x_3\ x_4 \quad x_n)$$

The r cursive proc dure nov can b easily written for the ith stage as

$$F_i(x_i, x_{i+1}, \ldots, x_n) = F_{i-1}(x_1, x_2, \ldots, x_{i-1}^*, x_i, \ldots, x_n)$$

$$\min_{x_i \ldots x_n} F_i = \min_{\substack{x_{i+1} \ldots x_n \\ x_i = g_i(x_{i+1} \ldots x_n)}} [V_i(x_{i+1}, \ldots, x_n) + \min_{x_i} U_i(x_i, \ldots, x_n)]$$

$$i = 1, 2, \ldots, n$$

At the nth stage we obtain

$$\min_{x_n} F_n(x_n) = \min_{x_n} U_n(x_n)$$

Therefore x_n^* is a constant, i.e., g_n is a constant function. Substituting x_n^* in $g_{n-1}(x_{n-1})$, x_{n-1} can be obtained. Continuing this procedure of back substitution all the x_i, $i = 1, 2, \ldots, n$ are obtained.

It is not difficult to see that the foregoing approach is very similar in spirit to the Gaussian elimination method of solving linear simultaneous equations which involves forward elimination and back substitution. The collection of functions

$$x_1^* = g_1(x_2, x_3, \ldots, x_4)$$
$$x_2^* = g_2(x_3, x_4, \ldots, x_n)$$

$$x_i^* = g_i(x_{i+1}, x_{i+2}, \ldots, x_n)$$

$$x_n = g_n \text{ constant}$$

is reminiscent of th upper triangular m tri which results after forward elimination In view of the foregoing observation we wish to call the recursiv algorithm dev loped above a g neralized Gaussiar elimination procedure

In the preceeding algo ithm we implicitly assumed that the indic ted single variable minimization is lways possible That is $\min_{y} U_1(x_1 \quad x_n)$ e ist for all i and f xed $x_{i+1} \quad x_n$ $i = 1^1 2 \quad n$ Ho ever this may not alway b possible for an arbitrarily specifi d nonlinea fur tion In addition single variable minimi ation itself may be difficult to carry out N v rtheless this approach has several advantages in the case of the ML-deconvolution problem ormulated in Sec 2 4 Before discussing that case we wish to cons d r the probl m of minimizing a quadratic func ion of n rea variables as an illustra- ion of the oregoing recursive algorithm

3 1 1 Real Variable Case

For the sake of simplicity we consider n = 3 Generalization to n variables is straightforward Let

$$F_1(x_1 \; x_2 \; x_3) = -2 \sum_{i=1}^{3} x_i b_i + \sum_{i=1}^{3} \sum_{j=1}^{3} x_i x_j a_{ij} + \sum_{i=1}^{3} r_i^2$$

$$\text{and} \quad a_{ij} = a_{ji} \quad \text{for all } i \; j$$

where b_i a_i and r_i $i = 1\ 2\ 3$ are constrants Carrying out the decomposition of $_1$ with respect to x_1 w obtain

$F_1 = U_1(x_1 \; x_2 \; x_3) + V_1(x_2 \; x_3)$

where $U_1(x_1 \; x_2 \; x_3) = -2x_1 b_1 + 2x_1 \sum_{j=2}^{3} x_j a_{1j} + x_1^2 a_{11}$

x_1^* which minimizes U_1 with x_2 x_3 fixed is given by

$$x_1 = (b_1 - \sum_{j=2}^{3} x_j a_{1j})/a_{11}$$

we assume that $a_{11} \neq 0$ Substituting x_1^* in F_1 we obtain

$$F_2(x_2 \; x_3) = -2 \sum_{i=2}^{3} x_i b_i + \sum_{i=2}^{3} \sum_{j=2}^{3} x_i x_j a_{ij} + \sum_{i=2}^{3} r_i^2 + (r_1)^2$$

where $b_i = b_i - b_1 \; a_{1i}/a_{11}$ $\quad i = 2 \; 3$

$$(r_1)^2 = r_1^2 - b_1^2/a_{11}$$

and $\quad a_{ij} = a_{ij} - a_{1i} \; a_{1j}/a_{11} \qquad i \; j = 2 \; 3$

Now $F_2(x_2 \; x_3) = U_2(x_2 \; x_3) + V_2(x_3)$

where $U_2(x_2 \; x_3) = -2x_2 b_2 + 2x_2 x_3 a_{23} + x_2^2 a_{22}$

therefore $\quad x_2^* = (b_2 - x_3 a_{23})/a_{22} \qquad a_{22} \neq 0$

using x_2^* we obtain $F_3(x_3) = -2x_3 b_3 + x_3^2 a_{33} + (r_1)^2 + (r_2)^2 + r_3^2$

minimization of F_3 gives

$$x_3 = b_3/a_{33} \qquad a_{33} \neq 0$$

Back substitution of x_3^* gives x_2^* which in turn gives x_1^*

The foregoing example clearly shows that the recur ive algorithm for minimization of a quadratic function or n variables is equivalent to Gaussian elimination technique for solving a set of linear equations

3 1 2 Discrete Variable Ca e

Havi g di cu ed certain aspects of the multivariable minimization with real variables we now conside the case when he var ables take values from a inite et Without any loss of generality we consider the quadratic func ion used in the p evious section with the d ferenc that th variables take values from the set $S = \{-3\ -2\ -1\ 0\ 1\ 2\ 3\}$ Extension to n-variable case and any given finite alph o t is straightforward

Following the steps as in the real variable cas in the first stage we have to find $\min_{x_1} U_1(x_1\ x_2\ x_3)$ In this case differentiation i not possible Hence we must tabulate values of x_1 which minimize U_1 for fixed x_2 and x_3 The tabulation format is shown in Table 3 1 No of en ries in th first column of the table will be 7^2 corresponding to 7^2 possible values of x_2 and x_3 Moreover for each set of x_2 and x_3 v u t ch ck 7 v lu s of x_1 o decide x_1^* Thus th first stage itself in vol s 7^3 evalu tions of U_1 Furth r in general it is not possible to expr s $_1^*$ in a closed form as in the case of i l variables Therefore x_1 can not be uostituted in F_1 to ob ain F_2 This implic that we must use h t ble for x_1^* in th second stage In the second stage we write F_2 as

$$F_2(x_2\ x_3) = U_1^*(x_1\ x_2\ x_3) + U_2(x_2\ x_3) + V_2(x_3)$$

Tabl 3 1 Stage 1

x_2	x_3	x_1^*	U_1
3	-3	-	-
3	-2	-	-
3	-1	-	
3	3	-	

Tabl 3 2 Stage 2

x_3	x_2	U_2
-3	-	-
-2	-	-
-1	-	-
0		
1	-	-
2	-	-
3	-	

For each value of x_2 and x_3 we can use U_1 from the table for x_1^* Thu in the second stage we minimize U_1+U_2 $(x_2\ x_3)$ with respect to x_2 by fixing x_3 There are 7 different values for x_3 and for each x_3 x_2^* is to be decided on the basis of 7 possible values of x_2 Thu the second table will have 7 entries as shown in Table 3 2 In a similar fashion we can carry out the computation required in stage 3 with the help of Table 3 2 of stage 2

Let us now examine the computational complexity for n-variable case with each variable taking q discrete values In the first stage q^n evaluations of U_1 is required Likewise in the ith stage q^{n-i+1} evaluation of $U_i+U_{i-1}^*$ is required On the other hand a direct search method would require q^n evaluation of

th funct on F It thus becomes clear that for the scheme described to become computa ionally efficient the unctional forms of U_1 U_2 U_n must be much impler than the functional form of F In otherword U_1 U_2 U_n must have the following desirable properties

1 U_1 s must be functions of a mall number of variables compared to the total number of variables involv d in the minimization problem

2 U_1 s must be easily computabl By this we mean the number of addition and multiplications needed for evaluating the function must be small

When both the propertie are sat sfied by an objective func ion it i possible to reduce the number of computations and storage requirements substantially using the foregoing recursive algorithm compared to the direct enumeration approach It may how ver be noted that storage requirement is still large compared to the enumeration approach

The obj ctive function in the cas of ML-deconvolution probl m formulat d in Section 2 4 is a special type of quadratic function called banded Toeplitz quadratic function Such a quadratic function me ts th above mentioned requirements In the next section we derive a ML-RDA or this case

3 2 RECURSIVE ALGORITHM FOR BANDED TOEPLITZ QUADRATIC FUNCTIONS

We recall from Section 2 4 that the function

$$F(x_1 \quad x_2 \qquad x_m) = 2 \sum_{i=1}^{m} x_i b_i + \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j a_{|i-j|} + c$$

where $a_{|i-j|} = 0$ for $|i-j| > g$ $g < m$

is called banded Toeplitz quadratic function (BTQF) for the matrix formed by $a_{|i-j|}$ as elements is a banded Toeplitz matrix

Let $F_1(\underline{x}) = F(x)$ r the sake of uniformity F_1 can be decomposed as

$$F_1 = -2x_1b_1 + 2x_1 \sum_{i=2}^{g+1} x_i a_{|1-i|} + x_1^2 a_o$$

$$+ [-2 \sum_{i=2}^{m} x_i b_i + \sum_{i=2}^{m} \sum_{j=2}^{m} x_i x_j a_{|i-j|}]$$

$$= U_1(x_1 \; x_2 \qquad x_{g+1}) + V_1(x_2 \; x_3 \qquad x_m)$$

Note that U_1 is only a function of g+1 variables Further U_1 is a function of consecutive g+1 variables rather than g+1 arbitrary variables as it may happen in the case of a general objective function This is a special property of BTQF In addition the form of U_1 at every stage is same Thus the same type of computation procedure is used in every stage Through the following example we have tried to illustrate various aspects of the algorithm The example is from [11]

Example 3 1

$$\text{Let } F_1(x_1\ x_2\ \quad x_7) = -2\sum_{i=1}^{7} x_i b_i + \sum_{i=1}^{7}\sum_{j=1}^{7} x_i x_j a_{|i-j|}$$

where $b_1 = 1\ 0$ $\quad b_5 = 1\ 5$ $\quad a_0 = 1$ $\quad a_1 = 0\ 5$ $\quad a_2 = -0\ 25$

$b_2 = 2\ 0$ $\quad b_6 = -3\ 0$ $\quad g = 2$

$b_3 = 0\ 5$ $\quad b_7 = 0\ 5$

$b_4 = 1\ 0$

and x_i $\quad i = 1\ 2 \quad 7$ take +1 or -1 values

Stage 1 $\quad F_1(x_1\ x_2 \quad x_7) = U_1(x_1\ x_2\ x_3) + V_1(x_2\ x_3 \quad x_7)$

$$U_1(x_1\ x_2\ x_3) = -2x_1b_1 + 2x_1\sum_{i=2}^{3} x_i a_{|1-i|} + x_1^2 a_0$$

$$= -3x_1 + 2x_1[0\ 5\ x_2 - 0\ 25x_3] + x_1^2$$

Table 3 3 is obtained by determining the value of x_1 that minimizes U_1 for a fixed x_2 and x_3

Table 3 3 Stage 1

x_2	x_3	x_1^*	U_1^*
-1	1	1	-5/2
-1	1	1	-3 5
1	1	1	-0 5
1	1	1	-1 5

Since $x_1 = 1$ for all the values of x_2 x_3 we can decide at this stage itself that the first variable is $x_1^* = 1$

Stage 2 $$U_2(x_2\ x_3\ x_4) = -2x_2 b_2 + 2x_2 \sum_{i=3}^{4} x_i a_{|2-i|} + x_2^2 a_0 + U_1(x_2\ x_3)$$

As the calculations are identical in all the stages we have presented the final results in Tables 3 4-3 8 corresponding to stages 2 to 6

Table 3 4 Stage 2

x_3	x_4	x_2^*	U_2
-1	-1	1	-4
1	1	1	-5
1	-1	1	-3
1	1	1	-4

Therefore the second variable $x_2^* = 1$

Table 3 5 Stage 3

x_4	x_5	x_3	U_3
-1	-1	1	-3 5
-1	1	1	-4 5
1	-1	1	-4 5
1	1	-1	-5 5

Table 3 6 Stage 4

x_5	x_6	x_4^*	U_4^*
-1	-1	1	-6
-1	1	1	-7
1	-1	1	-3
1	1	1	-4

Ta le 3 7 Stage 5

x_6	x_7	x_5^*	U_5
-1	-1	1	-7
-1	1	-1	-6 5
1	-1	-1	10 5
1	1	1	9 5

Table 3 8 Stage 6

x_6	x_7	U_6
1	-1	9 5
1	1	-12 5
1	-1	-2 5
1	1	-1 5

From Tabl 3 8 we th t x_6 = 1 and x_7 = 1 On back ubstitution Table 7 giv s x_5 = 1 It may ls b not c hat Table 3 6 nd Tabl 3 7 or stage 4 and 5 r spectively directly indicate hat x_4^* = 1 and x^* = -1 Using the values of x_4^* and in Table 3 we obta n x_3 = -1

5 5 VITERBI LGOPITHM FOR MINIMIZING QF

For the purpos of co par son we hav given a bri f des cription of VA in this section

R call that the QF min mization p oblem arose in conn ction with tran is ion of digit l data through a finite impulse respons hannel modelled as finite emo y transversal filter

At the kth instant he output of su h ilt r can be repr sented in terms pu x_k x_{k-1} x_{k-g} as

$$)_k = \sum_{1=0}^{q} h_1 x_{k-1} \qquad k - 1\ 2\ 3$$

wh r g is th memory o l transvcr al filter Thus y_k i a functioi of the pi sent input as well a the previous state of h transversal filtcr Lct S_k denote the tate of the transversal filt r at th k th ii tan Th i

$$S_k = (x_k\ x_{l-1} \qquad x_{k\ g+1}) \qquad x_k = 0 \quad \text{if } k < 0$$

In t rms of S_k s y_l can be expressed a

$$y_k = f(h_o\ h_1 \qquad h_g\ S_k\ S_{k-1})$$

Th icfoic given an inpu sequence $\{x_k\}$ we have a corresponding st t sequence $\{S_k\}$ and an output sequence $\{y_k\}$ This co respondence is one to one This observation i crucial to the development of VA Due to this corr spondence we can minimize $F(\underline{x})$ ov r S_k s instead of x_k s For this purpose decompo c $F(\underline{x})$ as

$$F(x) = f_1(\underline{x}) = [-2 \sum_{1=2}^{n} x_1 b_1 + \sum_{1=2}^{m} \sum_{j=2}^{n} x_1 x_j a_{|1-j|}] +$$

$$[-2x_1 b_1 + 2x_1(x_2 a_1 + x_3 a_2 + \qquad x_{g+1} a_g) + x_1^2 a_o]$$

Note that the quantity in ide the second pair of brackets is only a function of $x_1\ x_2 \qquad x_{g+1}$ Le

$$f_1(\underline{x}) = u_1(x_1 \qquad x_{g+1}) + v_1(x_2 \qquad x_m)$$

It is easy to see that $u_1(x_1, x_2, \ldots, x_{g+1}) = u_1(S_{g+1}, S_g)$. Now v_1 can be similarly decomposed as

$$v_1(x_2, x_3, \ldots, x_m) = v_2(x_3, x_4, \ldots, x_m) + u_2(S_{g+2}, S_{g+1})$$

In a similar fashion we can write for the kth stage

$$v_k(x_{k+1}, \ldots, x_m) = v_{k+1}(x_{k+1}, \ldots, x_m)\ u_k(S_{g+k}, S_{g+k-1})$$

$$k = 1, 2, 3, \ldots, m-g$$

Therefore $F(\underline{x})$ can be written as

$$F(S_1, S_2, \ldots, S_m) = \sum_{k=1}^{m-g} u_k(S_{g+k}, S_{g+k-1}) + v_{m-g}(S_m)$$

where

$$u_k(S_{g+k}, S_{g+k-1}) = -2x_k b_k + 2x_k \sum_{i=k+1}^{g+k} x_i a_{|k-i|} + x_k^2 a_0$$

$$k = 1, 2, 3, \ldots, m-g$$

We wish to note that the derivation of the foregoing recursive equations is based on the approach given by Hayes [41]. The optimality of the algorithm follows from Lemma 3.1.

The algorithm works as follows. In the first stage of the algorithm $u_1(S_{g+1}, S_g)$ is minimized with respect to S_g for fixed S_{g+1}. As S_{g+1} can assume anyone of the q^g states we have to store u_1^* value corresponding to q^g states. As S_g and S_{g+1} differ in only one place determination of optimum S_g given S_{g+1} is effectively one variable optimization. To complete our discussion let

$$F(S_g \quad S_m) = f_1(S_g \quad S_1) = u_1(S_{g+1}\ S_g) + \sum_{k=2}^{n-g} u_1(S_{g+k}\ \ S_{g+k-1}) + v_{m-g}(S\ )$$

Hence

$$\min_{S_g\ S_m} f_1 = \min_{S_{g+1}\ \ m} \left[\sum_{k=2}^{n-g} u_k(S_{g+k}\ S_{g+k-1}) + v_{m-g}(S_m) + \min_{S_g} u_1(S_{g+1}\ \ S_g)\right]$$

Define $u_k^* = u_k(S_{g+1}\ \ S_{g+l-1})$ $\quad u_o^* = 0$

Th n th recur i /e e uations for /A an b written as

$$u_k^* = \min_{S_{g+k-1}} [u_k(S_{g+l}\ \ S_{g+k-1})\ \ u_{k\ 1}] \quad k - 1\ 2 \qquad m-g$$

Finally the ootimum value of F F i g ven as

$$F^* = \min_{S_m} [u_{m-g} + v_{m\ g}\ (S_m)]$$

Result obtained by applying VA to E ample 3 1 is shown in he Figure 3 1 In t e ficure we have shown the survivor paths hrough the trelli In h figure dotted lines indicate alter- nate survivors giving same function values From the figure it may be noted that (1 1) i node common to all the survived paths in stage 3 Th s nplies that $x_1 = 1$ Thus decisions regarding data can be made ven before completing the forward recursion process This is due to the merging of the urvivor o ths through the Trellis

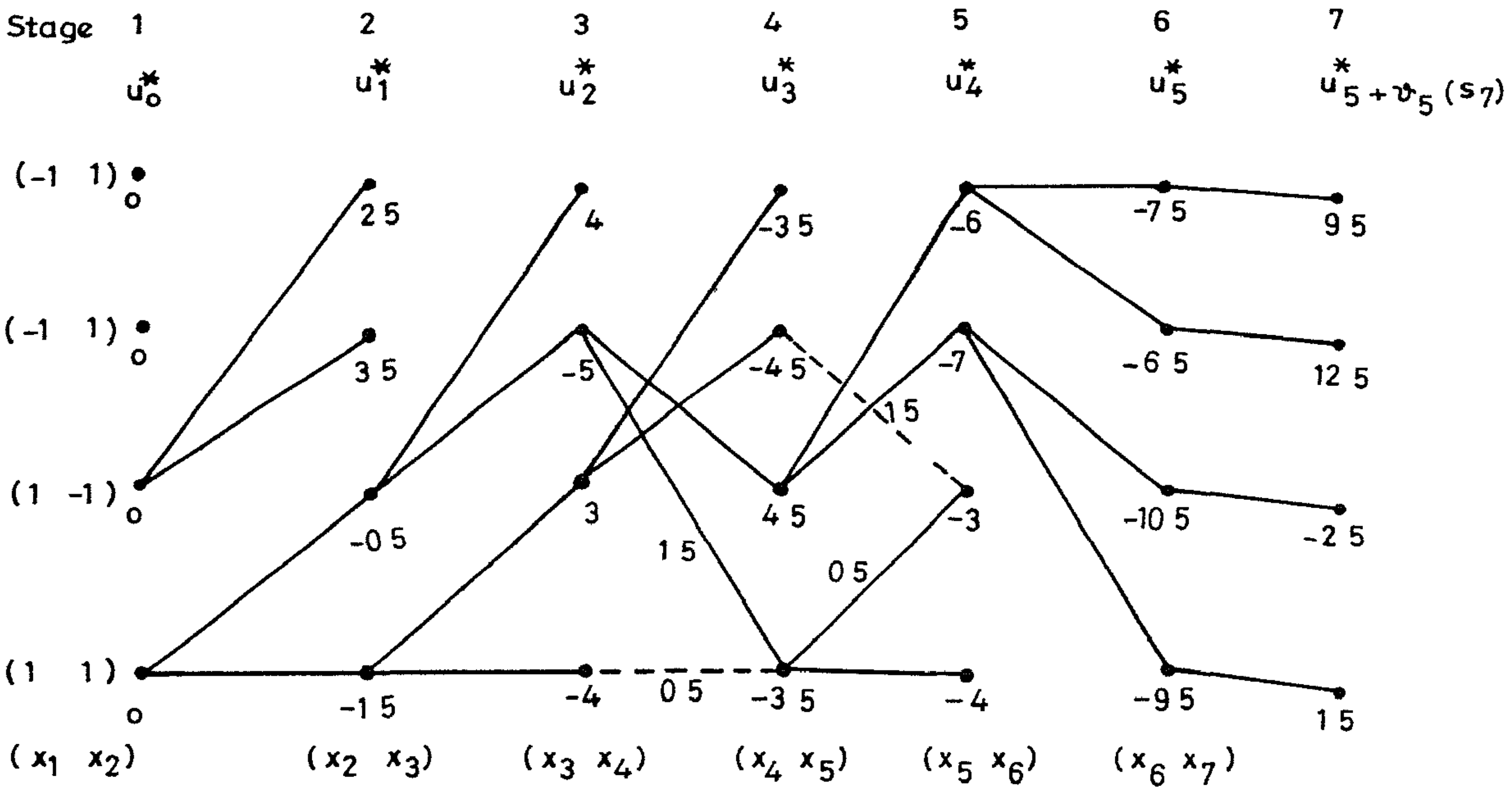


FIG 3 1 Trellis diagram showing survivor paths

Let us now consider the solution obtained by applying ML-RDA to Example 3 1 From the Tables 3 3-3 8 w can easily see that $x_1 = 1$ Similarly table given for stage 2 gives $x_2^* = 1$ The third stage shows that $x_3 = -x_4^*$ Fourth stage gives $x_4^* = 1$ and fifth stage $x_5 = -1$ These aspect do not become clear from the Trellis diagram shown in Figure 3 1

3 4 TRATECIES FOR REDUCING COMPUTATIO AL AND STORAGE REQUIREMENTS

Througrout the discussion of th recursiv deconvolution algorithm we have used tables to how the results obtained at each stage of the algorithm In actual practice su h tables are not necessary Notice that the table used in the first stage is used only by the second stage Therefore in the second stage we can use the updated v rsion of the first table For the purpose of illustration w have shown the development of the tables for Example 3 1 in Figure 3 2 Back substitution gives the path snowing dotted lines in he figure It is obtained as follows x_6 and x_7^* ar decided by noting that U_6^* is min mum at that point x_5 for the state $(x_6^* \; x_7) = (1 \; 1)$ is -1 x_4^* for the state $(x_5^* \; x_6) = (-1 \; 1)$ is 1 x_3 for the state $(x_4 \; x_5) = (1 \; -1)$ is -1 x_2^* and x_1^* are obta ned similarly

As the procedure is same for a general n-variable BTQF we see that the storage requirem nt is q^g registers of length (m+1) ach for storing x_1^* s at each stage and the accumulated minimum

Stage 1

x_1	x_2	y_3	U_1^{+}
1	-1	-1	-5/2
1	-1	1	-3 5
1	1	-1	-0 5
1	1	1	-1 5

Stage 2

x_1^*	x_2	x_3	x_4	U_2
1	1	-1	-1	-4
1	1	-1	1	-5
1	1	1	-1	-3
1	1	1	1	-4

Stage 3

x_1^*	x_2^*	x_3^*	x_4	x_5	U_3^*
1	1	1	1	-1	-3 5
1	1	1	-1	1	-4 5
1	1	1	1	-1	-4 5
1	1	-1	1	1	-3 5

Stag 6

x_1^*	x_2	x_3^*	x_4^{+}	x_5^*	x_6	x_7	U_6^{+}	State	
1	1	1	1	-1	-1	-1	-9 5	-1	-1
1	1	1	1	-1	-1	1	-12 5	-1	1
1	1	-1	1	-1	1	-1	-2 5	1	-1
1	1	-1	1	-1	1	1	-1 5	1	1

Figure 3 2 Developm nt of the tables in ML-RDA for Example 3 1

value at every stag Thus the storage requirement grows linearly with the blo k length Further since at every stage e need q^g function evaluations the computational complexity of the lgorithm grows linearly with block length and exponentially with ch nnel me ory

Even with the foregoing strategy for storing the tables t e alg rithm still re uires a large amount of storage and computations for even mod rate block lengths Thus if th recursive algorithm is to b come acceptable it must hav additional features for minimising storage and computational requirements Towards this end w now propose and examine a method for reducing the complexity of ML-RDA which is expected to re ult in a greater savings in storag and computations

To b gin with let us examine Tables 3 3-3 8 for Exampl 3 1 For instance consider Table 3 5 obtained at the 3rd stage Note that if we are able to xpress x_3^* as a function of x_4 and x_5 substi uting for x_3^* in the objective function eliminates the need for the table in th n xt stage computations Moreover for the purpo e of b ck substitu ion we need retain only the functional form of x_3 In this case it is easy to see that $x_3^* = -x_4$

If at every stage we are able to obtain such simple r present tion stor ge requir ments are reduced to a minimum In the example under consideration all the functions have simple representations as given below

tag	1	2	3	4	5	6	
Functions	$\overset{*}{x}_1=1$	$y_2=1$	$x_3^*=-x_4$	$x_4=1$	$x_5^*=-1$	$x_6=1$	and $x_7=1$

Su h i nple functional forms canrot always be guaranteed For the t re being s ume that such representations ar possible Then the next questi n would be c in we arrive at the funct onal form without numerating all possible values of the fix d vari l s? If this i possible w would also h v achieved reduction in the computational co plexity

However in g n ral this ppears to be a difficult problem In the fir t plac functional form of x_1 may be complicat d due to which storag f uch func on as well a substitution int the objectiv function may b difficult Secondly it may be difficult to directly obt r the functional forms Therefore to begin with instead of considering the most general ca c w study th binary case in th next sec ion

4 1 Binary Data Transmission

Wh n the data symbols assume on of th two values 1 or -1 the objective unction can be converted into another function called ps udo oolean function (PBF) [57] which is a function of Boolean variables with re l coc ficients For this we us the tr nsformati n $y = (1+x)/2$ (which mc ns y takes 0 nd 1 as x takes -1 and 1 resp ctively) With the help of this transformation the qu dratic function $F(\underline{x})$ can be convert d into a PBF $F(\underline{y})$ wher y_1 y_2 y_m are Boolean variables

Le $F(\underline{x})$ expressed in the matrix vector form as

$$F(\underline{x}) = -2x^T b + x^T A\underline{x} + \underline{r}^T \underline{r}$$

We use th transformation $\underline{y} = (\underline{1}+x)/2$

where $\underline{1}$ is a vector of all ones With this t an formation we obtain

$$F(\underline{y}) = -2\underline{y}^T \quad + \underline{y}^T A\ \underline{y} + p$$

where $b = 2(\underline{b}+A\underline{1})$ $A = 4A$ $p = 2 1^T b + \underline{1}^T A 1 + \underline{r}^T r$

The con tant term p can be ignored for he purpose of minimizing $F(\underline{y})$

$F(\underline{y})$ is same as $F(\underline{x})$ Therefor the ML-PDA given in S ction 3 2 applies to this case with the difference that the tables at each tage now contains Boolean variables As our interest is in xpressing x_i^* in terms of x_{i+1} x_{i+2} x_{i+g} at the ith stage we observ that the table in this case is a truth table for x_i^* in terms of x_{i+1} x_{i+2} x_{i+g} Now using re ults from the theo y of Boolean functions [72] we can represent x_i in some convenient form For ins ance minimal sum of produ t or product o sums form

Thus we have shown that in the case of binary data there exists a procedure for expre sing x_i^* in t ms of x_{i+1} x_{i+2} x_{i+g} in the ith stage However simple expressions for x_i^* may not always be obtained Let us consider the worst case where x_i^*

is rcor sented in terms of x_{i+1} $x_{i\ g}$ The truth tabl or x_i^* will contain 2^g ntries   In this case h stor ge can bc reduced to some extent by obta_ning a mi imal representa tion of x_i t thods such as Quin-McClausky or any such similar m thod [72] could oe used for this puipose   However we may have to use a special purpo c computer which implement Boolean function minimization algor thm

Now we cons der the probl m of obtaining unctional repre-senta ion for x_i^* withou enum ating all po ible 2^g cases   By using an algorithm d veloped by Hammer et al [57] we show that the computational complex ty can bc reduced at very stage without sacrifying optimality   Towards this end we use L mma 3 1 to derive a r cursive algorithm developed originally by Hammer et al [57] for minimizing nonlinea PBF s   in the next section   The approach taken here i much simpler compared to the derivation given in [57]

3 4 2 Minimization of Pseudo-Boole n Functions

Considcr a PBF $F_1(x_1\ x_2 \qquad x_m)$ of m variables   Since pseudo-Boolean function is linear in each of its variables we can write

$$F_1(x_1\ x_2 \qquad x_m) = x_1 U_1(x_2\ x_3 \qquad x_m) + V_1(x_2\ x_3 \qquad x_m)$$

Application of Lemma 3 1 gives

$$\min_{x_1 \quad x_m} F_1 = \min_{x_1 \quad x_m} [V_1(x_2\ x_3 \quad x_m) + \min_{x_1} x_1 U_1(x_2\ x_3 \quad x_m)]$$

Now since x_1 s are Boolean variables it follows that

$$x_1 = \begin{cases} 0 & \text{if} \quad U_1(x_2\ x_3 \quad x_m) > 0 \\ \emptyset & \text{if} \quad U_1(x_2\ x_3 \quad x_m) = 0 \\ 1 & \text{if} \quad U_1(x_2\ x_3 \quad x_m) < 0 \end{cases}$$

where $\emptyset$ is a don t care variable (x_1 arbitrary)

Therefor a Boolean function representation for x_1 can be obtained by determining those $x_2\ x_3 \quad x_m$ for which $U_1 < 0$ Substitution of x_1^* into F_1 gives F_2 The other stages of the minimization follow the same pattern as in the case of recursive procedure described earlier Thus at the ith stage we have

$$\min_{x_i \quad x_m} F_i = \min_{x_{i+1} \quad x_m} [V_i(x_{i+1}\ x_{i+2} \quad x_m) + \min_{x_i} x_i U_i(x_{i+1}\ x_{i+2} \quad x_m)]$$

and

$$x_i^* = \begin{cases} 0 & \text{if} \quad U_i > 0 \\ \emptyset & \text{if} \quad U_i = 0 \\ 1 & \text{if} \quad U_i < 0 \end{cases}$$

As we are interested in only one solution $\emptyset$ can be fixed as 0 or 1 and we can solve $U_1 \leq 0$ or obtaining x_1 or $U_1 \geq 0$ for obtaining x_1^* Hammer et al [57] have also developed an algorithm or determining all solutions which satisfy $U_1 \leq 0$ This algorithm can be used to reduce the computational complexity a each stage

From the foregoing discussion it is clear that in the binary case it is possible to r duce storage and computational complexity of the RDA if fast processors are available for minimization of Boolean functions and for implementing Hammer t al procedure for obt ining x_1

It may be noted that determination f minimal representation of Boolean functions is a classical problem in the theory of switching functions Although a large number of methods are available [72] no definite claim can be mad in favour of one approach over the other

In summary the over ll effici ncy of the RDA if impl mented in the manner described in this ection depends to a great extent on the availability of fast algorithms for obtaining minim l repres ntations for x_1^* either in the Boolean func ion form or in the pseudo-Bool an function form

In the nonbinary case however no such claims can be made This aspect needs furth r investigation

3.5 RECURSIVE ALGORITHM FOR DECODING BLOCK ERROR CORRECTING CODES

As an application of the recursive algorithm developed in the previous section we consider maximum likelihood decoding of binary block codes. Our approach consists in formulating the ML decoding problem as an equivalent problem of minimizing a pseudo-Boolean function of several variables. It may be noted that the algorithm developed in this section can also be used for maximum likelihood decoding of convolution codes.

We wish to note that Wolf [73] has studied the application of VA to decode block codes by constructing Trellis diagrams. The approach considered here does not require Trellis construction and it seems to provide a more general method for maximum likelihood decoding of binary block codes.

The codes considered here are over GF(2), where GF(2) denotes Galois field with two elements 0 and 1 [72]. For such codes the ML decoding is equivalent to solving the set of linear equations $G\underline{x} = r$ over GF(2) such that the Hamming distance between $G\underline{x}$ and r is minimum when all the code words are assumed to equiprobable. The matrix G is called the generator matrix of the code in the parlance of coding theory [6] [7]. The received vector $\underline{r}$ is an additive combination of the transmitted codevector which is not observable at the receiver and the noise vector $\underline{w}$. That is $\underline{r} = \underline{c} \oplus \underline{w}$ where $\oplus$ denotes addition

operation in GF(2) It may be notcd that the lements of $\underline{r}$ and $\underline{w}$ are e ther 0 or 1 We also assume that G is an nxm m trix

The Hamming distance between two vectors $\underline{x}$ and $\underline{y}$ over GF(2) is defined as the number of nonzero entri s in $x \oplus \underline{y}$ It is also known as the weight of $\underline{x} \oplus y$ d noted as $w(\underline{x} \oplus \underline{y})$

Since GF(2) elements are also elements of the real field we can show that

$$\underline{p} \oplus \underline{q} = \underline{p} + \underline{q} - 2\underline{p}\ \underline{q}$$

where $\underline{p}\ \underline{q}$ denote pointwise multiplication We now make use of the property that if y s 0 1 real vector then

$$w(\underline{y}) = \underline{1}^T \underline{y}$$

where $\underline{1}$ is column vector of all ones Therefore it is easy to see that

$$w(\underline{p} \oplus \underline{q}) = w(\underline{p}+\underline{q}-2\underline{p}\ \underline{q}) = w(p)+w(q)\ 2w(p\ \underline{q})$$

Thus maximum likelihood decoding problem can be formulated as

$$\min_{x}(w(G\underline{x})+w(r)-2w(G\underline{x}\ \underline{r}))$$

$$= \min_{x}(\underline{1}^T G\underline{x}+\underline{1}^T \underline{r}-2\underline{1}^T(G\underline{x}\ \underline{r})) \qquad (3\ 5)$$

Note that $G\underline{x}$ is to be computed over GF(2). Therefore we need a suitable representation for the elements of $G\underline{x}$. Let $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ denote n rows of G. Then we can write $G\underline{x}$ as $(\underline{a}_1\underline{x}, \underline{a}_2\underline{x}, \ldots, \underline{a}_n\underline{x})$ where $\underline{a}_i\underline{x} = a_{i1}x_1 \oplus a_{i2}x_2 \oplus \ldots \oplus a_{im}x_m$. We can use the relation

$$u \oplus v = u+v - 2uv \qquad (3.6)$$

in order to convert $\underline{a}_i\underline{x}$ into an equivalent pseudo-Boolean function. However, such a representation becomes too complicated when more variables are involved. A more compact form of representation is necessary to obtain a representation which reflects the generator matrix explicitly. Towards that end consider the transformation $u' = 1-2u$ and $v' = 1-2v$. Substituting for u and v on the right hand side of (3.6) we obtain $u \oplus v = (1-u'v')/2$ which is a more compact expression compared to u+v-2uv. We can further show that

$$u \oplus v \oplus w = (1-u'v'w')/2 \quad \text{where} \quad w' = 1-2w$$

In general we have $\sum_{i=1}^{m} c_i x_i = (1 - \prod_{i=1}^{m} z_i^{c_i})/2$ where Σ denotes summation in GF(2) and $c_i \in GF(2)$, $i = 1, 2, \ldots, m$ and $z_i = 1-2x_i$. Now for $G\underline{x}$ we have

$$\underline{a}_i\underline{x} = \frac{1}{2}\,(1- \prod_{j=1}^{m} z_j^{a_{ij}}) \quad i = 1, 2, \ldots, m$$

$$z_j^{a_{ij}} = \begin{cases} 1 & \text{if } a_{ij} = 0 \\ z_j & \text{if } a_{ij} = 1 \end{cases}$$

Note that the representation for $\underline{a}_i\underline{x}$ clearly shows the ffect of a_{ij} s

Going back to the minimization problem given by (3 5) we can ignore $\underline{1}^T\underline{r}$ for th purpos of minimization as it is a const nt The rest of the terms can be combined as

$$\min_{x} \; (\underline{1}^T G\underline{x}\ \underline{r}) \quad \text{where} \quad \underline{r} = \underline{1}-2\underline{r}$$

Let $\underline{d} = (d_1\ d_2 \qquad d_n)^T$ where $d_j = \prod_{i=1}^{m} z_i^{a_{ij}}$ $j = 1\ 2 \qquad n$

Then $G\underline{x} = \frac{1}{2}(1-\underline{d})$ Therefore $G\underline{x}\ r = \frac{1}{2}\underline{1}\ r - \frac{1}{2}\underline{d}\ \underline{r}$

Thus we have to find

$$\min_{\underline{z}} \left(-\frac{1}{2}\underline{1}^T(\underline{d}\ r)\right) = \min_{\underline{z}} \left(-\frac{1}{2}\sum_{j=1}^{n} d_j r_j\right) \quad \text{where } r_j = 1-2r_j$$

With the objective function in this form we can now proceed with the recursive optimization algorithm discussed in Section 3 4 1 To round up the foregoing discussion we give thre illustrative examples

Example 3 2 (Hamming Code) Consider the generator matrix G given by

$$G^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $\underline{x} = (1\ 1\ 1\ 0)^T$ be the transmitted vector Then

$\underline{c} = (1\ 1\ 1\ 0\ 0\ 0\ 0)^T$ is th transmittcd code v ctor

L t $\underline{r} = (0\ 1\ 1\ 0\ 0\ 0\ 0)^T$ be the received vector For this situa ion we have

$d_1 = z_1 \quad d_2 = z_2 \quad d_3 = z_3 \quad d_4 = z_4 \quad d_5 = z_1 z_2 z_4$

$d_6 = z_1 z_3 z_4$ and $d_7 = z_2 z_3 z_4$

Therefor the function to be minimized is given by

$$f(\underline{z}) = -\frac{1}{2}\,(z_1 - z_2 - z_3 + z_4 - z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4)$$

Negl cting 1/2 we can write $f(\underline{z})$ as

$$f(z) = z_1(-1 - z_2 z_4 - z_3 z_4) - z_2 - z_3 + z_4 + z_2 z_3 z_4 =$$

$$U_1(z_1\ z_2\ z_3\ z_4) + V_1(z_2\ z_3\ z_4)$$

Minimization of U_1 with r spect to z_1 for fixed z_2 z_3 and z_4 giv s the following table for stage 1

Stage 1

z_2	z_3	z_4	z_1^*	U_1^*
-1	-1	-1	1	-3
-1	-1	1	1	-1
-1	1	-1	1	-1
-1	1	1	1	-1
1	-1	-1	1	-1
1	1	1	1	-1
1	1	-1	-1	-1
1	1	1	1	-3

Stage 2

z_3	z_4	z_2^*	U_2^*
-1	-1	-1	-3
-1	1	1	-3
1	-1	-1	-3
1	1	1	-3

In terms of U_1^* we can write f(z) as $f(\underline{z}) = U_1^*(z_2\ z_3\ z_4) + z_2(1-z_3z_4) + z_3-z_4$

Minimization of $f(\underline{z})$ with resp ct to z_2 gives the table for stage 2 Note th t U_2^* is independent of z_3 and z_4 Therefore $f(z) = -3+z_3-z_4$ Minimizing finally with respect to z_3 and z_4 we obtain

$f(\underline{z}) = -5$ when $z_3^* = -1$ and $z_4 = 1$ which on back substitution gives $z_2^* = -1$ and $z_1 - -1$ Therefore we obtain (1 1 1 0) as the recovered data vector

Example 3 3 (Reed Muller Cod) Consider the generator matrix given by

$$G^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $\underline{x} = (1\ 1\ 1\ 1)^T$ be the transmitted vector

$\underline{c} = (1\ 0\ 0\ 1\ 0^*\ 1\ 1\ 0^*)$ and $\underline{r} = (1\ 0\ 0\ 1\ 1\ 1\ 0\ 0)^T$

In this case the function to be minimized is given by

$$f(\underline{z}) = -\frac{1}{2}\left[-z_1+z_1z_2+z_1z_3-z_1z_2z_3-z_1z_4-z_1z_2z_4+ z_1z_3z_4+z_1z_2z_3z_4\right]$$

$f(\underline{z})$ can b written as

$$f(\underline{z}) = \frac{z_2}{2}(-z_1+z_1z_3+z_1z_4-z_1z_3z_4)+\frac{1}{2}(z_1-z_1z_3+z_1z_4\ z_1z_3z_4)$$

For this case the table obtained for the first stage is shown below

tage 1

z_1	z_3	z_4	z_2^*	U_2
1	-1	-1	-1	-2
-1	-1	1	∅	0
-1	1	-1	1	-1
-1	1	1	∅	0
1	-1	-1	1	-2
1	-1	1	∅	0
1	1	-1	∅	0
1	1	1	∅	0

$$U_2^*(z_1\ z_3\ z_4) = \min_{z_2} \frac{z_2}{2} (-z_1+z_1z_3+z_1z_4-z_1z_3z_4)$$

Note that in the first stage we have considered z_2 instead of z_1 z_1 also can be consid red as the f rst stage variable

From the tabl it is not difficult to s e that $z_2^* = z_1z_3z_4$ Substituting z_2^* in $f(\underline{z})$ we obtain

$$f_2(z_1\ z_3\ z_4) = \frac{z_1}{2} (1-z_3+z_4-z_3z_4) + \frac{1}{2} (-z_3z_4+z_4+z_3\ 1)$$

Tabl for th second stag minimization is given below

Stage 2

z_3	z_4	z_1^t	U_1
-1	-1	$\emptyset$	0
-1	1	-1	-2
1	-1	$\emptyset$	0
1	1	$\emptyset$	0

From the table it is easy to see that $z_1 = z_3$ Substituting for z_1 in $f_2(z_1\ z_3\ z_4)$ we obtain

$$f_3(z_3\ z_4) = z_3 - 1$$

Therefore we obtain $z_3 = 1$ $z_4^* = \emptyset$ $z_2^t = \emptyset$ $z_1^* = -1$

where $\emptyset$ d notes either 1 or -1 Thus the maximum likelihood solutions are

(-1 1 -1 1) and (-1 -1 -1 1)

The corresponding 0 1 vectors are (1 0 1 0) and (1 1 1 1) Thus although the transmitted vector is (1 1 1 1) we obtain (1 0 1 0) also as a solu ion This is because two errors were introduced in the code vector at the places indicated by * in $\underline{c}$ The same received vector $\underline{r}$ can be obtained by introducing

two rrors in the locations 2 and 4 of the code vector $\underline{c}_1$ obtained by (1 0 1 0) as shown below

$$\underline{c}_1 = (1\ 1\ \ 0\ 0\ \ 1\ 1\ 0\ 0)^T$$

Example 3 4 Con ider the generator matrix given by

$$G^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

d_j s can be obtained by reading columnwise in G^T and associating z_i s whenever 1 s are encountered For instance w obtain d_{16} by noting that the last column contains 1 in the z_6th position Therefore $d_{16} = z_6$ The objective function is then given by

$$f(z_1\ z_2\ \ \ \ z_6) = -\frac{1}{2}\sum_{i=1}^{16} d_i r_i = -\frac{1}{2}(-z_2-z_1z_2+z_1z_3+z_1z_2z_3-$$

$$z_2z_4-z_2z_3z_4-z_3z_5\ z_3z_4z_5-z_4z_6-z_4z_5z_6-z_5-$$

$$z_5z_6+2z_6)$$

where we assume that the transmitted vector is $(0\ 1\ 0\ 0\ 1\ 0)^T$

$$c = (0^*\ 0\ 1\ 1\ 0\ 1^*\ 1\ 1\ 1\ 1^*\ 0\ 1\ 1\ 1\ 0\ 0)^T$$

and $\underline{r} = (1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0{,}1\ 1\ 1\ 0\ 0)$

Note that 3 errors occur in this case

Now carrying out the steps of the recursive algorithm we obtain th following tables

Stage 1

z_2	z_3	z_1	U_1^*
1	1	1	1
1	-1	-1	-3
-1	1	1	1
-1	-1	1	-1

$U_1 = z_1(z_2-z_3-z_2z_3)$

Stage 2

z_3	z_4	z_2	U_2^*
1	1	1	-4
1	-1	1	-2
-1	1	$\emptyset$	-2
-1	-1	1	-2

$U_2 = z_2(1+z_4+z_3z_4)+U_1^*$

In stage 2 it is easy to note that $z_2^t = -z_3z_4$ if we put $\emptyset = 1$

Stage 3

z_4	z_5	z_3^*	U_3^*
1	1	-1	-4
1	-1	1	-6
-1	1	$\emptyset_1$	-2
-1	-1	$\emptyset_2$	2

$U_3 = z_3(z_5+z_4z_5)+U_2$

Stage 4

z_5	z_6	z_4^*	U_4^*
1	1	-1	-4
1	-1	1	-6
-1	1	1	-6
-1	-1	1	-6

$U_4 = {}_4(\ {}_6+z_5z_6)+U_3^*$

$z_3^* = -z_4 z_5$ if $\emptyset_1 = 1$ and $\emptyset_2 = -1$

Stage 5

z_6	z_5	U_5
1	-1	-8
-1	$\cancel{1}$	6

$U_6 = -2z_6 + U_5^*$

$U_6^* = -10$

$z_6^* = 1$

$U_5 = z_5(1+z_6) + U_4$

Tracing back through the tables we obtain

$z_6 = 1 \quad z_5 = -1 \quad z \;= 1 \quad z_3 = 1 \quad z_2 = -1 \quad z_1^* = 1$

Therefore th maximum likelihood solution vector is $(0\ 1\ 0\ 0\ 1\ 0)^T$ which is the transmitted data vector

CHAPTEP 4

DECISION FEEDBACK DECONVOLUTION SCHEMES

In the previous chapter we derived a ML RDA whose computational complexity and stor ge requirements grow linearly with the block length and exponentially with th channel memory We showed that attempts to further alleviate computational ard storage requirements in th binary ca e lead to minimization of pseudo Boolean func ons which in general is a rather difficult problem The computational and storage requirements how ver can be reduced if one is willing to accept suboptimal ch mes Towards this end in this chapter we con ider the possibility of incorporating decision ecdback idea for recov ring data in BDI systems We propose and analyse four decision feedback data recovery (DF-DR) schemes All these s iemes are based centrally on the idea of nonlinear Gaussian elimination method described below

Consider the received vector r given by (2 7)

$$\underline{r} = H\underline{x} + w \qquad (4\ 1)$$

Under the assumption or r al valued data and white Gaussian noise the ML data vector is obtained by solving

$$H^T H x = H^T r \qquad (4\ 2)$$

One method of solving (4 2) is to use the Gaussian elimination (GE) algorithm which consists of forward elimination procedure (in which $H^T H$ is reduced to an upper triangular matrix) and back substitution (in which the solution is obtained by recursively solving th resulting set of equations) Now we can incorporate the discrete nature o data symbols into GE algorithm in the following manner While back substitutin instead of using the calculated valu s directly we can us the value obtained by passing the calculated valu th ough a nonlinear d cision device which assigns an appropriate valu from the finite alphabet Since this proc dure involve in roduction of a non linear device between the fo ward elimination and back substitu ion stages the method s called nonlin ar Gaussian elimina ion

This chapter is organized as follows In S ction 4 1 we us Cholesky decomposition t chnique to derive a DF-DR scheme which does not t e noise variance into account Section 4 2 gives a DF-DR sch me which minimizes mean s quare error between the actual data and its estimate A numerically stable DF DR sc me is derived in Section 4 3 This c eme is based on the minimum property of Fourier expansion in finite dimensional Hilbert spaces In Section 4 4 a DF-DR sch m is derived by modifying Austin s [13] assump ions regarding decision feedback equalizer for CSDT systems In S ction 4 5 we show that the modified Austin s pproach is also essen ially equivalent to a

nonlinear Gaussian elimination method In Section 4 6 we list some observations on the methodology used in this chapter for motivating the idea of decision feedback

4 1 DECISION FEEDBACK SCHEME USING CHOLESKY DECOMPOSITION

Cholesky decomposition technique [74] deals with decomposition of positive definite matrices into product of nonsingular lower and upper triangular matrices If V denotes a lower triangular matrix we can write $H^T H$ as

$$H^T H = V^T V \tag{4 3}$$

Using (4 3) (4 2) can be written as

$$V^T \underline{x} = V^{-1} H^T r$$

This is equivalent to the forward elimination step of GE algorithm Since V^T is an upper triangular matrix estimate of the last element of $\underline{x}$ is obtained as

$$\tilde{x}_m = c_n / v_{mm}$$

where c_m is the mth element of $c = V^{-1} H^T \underline{r}$ and v_{mm} is the mth diagonal element of V^T Now let $\hat{x}_m$ be the value obtained by passing $\tilde{x}_n$ through a decision device We use $\hat{x}_m$ instead of $\tilde{x}_m$ for calculating $\tilde{x}_{m-1}$ given by

$$\tilde{x}_{m-1} = (c_{m-1} \quad v_{m\ 1\ m} \hat{x}_m) / v_{m-1\ m-1}$$

Now $\hat{}_{m-1}$ w ll be t e re ov red data d cide l on the basis of $\tilde{x}_{n-1}$ Using $_m$ ana $\hat{x}_{m-1}$ $\tilde{x}_{n-2}$ is calculatec This proc dure is continucd till $\hat{x}_1$ is obtained

In the foregoing approach la t data o th block is re recover d first her as in h c s of dcci ion f dback data r cov ry sc mes fo CSDT first data is recovered first Al hough which da a value is recover o first i imm terial in t e case o BDT for the s le of uniformity we incorporate thi feature of CSDT systems in th following manner

$H^T H$ can also be written as a product of upper and lower triangular matrices That is

$$H^T H = U^T U \tag{4 4}$$

where U is the upper triang lar matrix One method o obtaining U is as follows

Let $A_m = H^T H$ and $A_{m\ 1} = U_{m-1} U^T_{m-1}$

Th n A_m and U_m can be written as

$$A_m = \begin{bmatrix} a_{mm} & d^T_{m-1} \\ d_{m-1} & A_{m-1} \end{bmatrix} \quad \text{and} \quad U_m = \begin{bmatrix} y & e^T_{m-1} \\ 0 & U_{m-1} \end{bmatrix}$$

Then

$$A_m = \begin{bmatrix} y^2 + e^T_{m-1} e_{m-1} & e^T_{m-1} U^T_{m-1} \\ U_{m-1} e_{m\ 1} & U_{m-1} U^T_{m-1} \end{bmatrix}$$

Therefore $y = (a_{mm} - \underline{e}^T_{m-1}\underline{e}_{m-1})^{1/2}$ and $\underline{d}_{m-1} = U_{m-1}\underline{e}_{m-1}$

Therefore $\underline{e}_{m-1} = U^{-1}_{m-1}\underline{d}_{m-1}$

Note that y is real because $\det(U_m U_m^T) = \det(U_{m-1})^2\, y^2$

$$= \det(A_m)$$

and A_m is a positive definite matrix. In deriving the recursive form for U_m we have followed the method given in [74] (see page 423).

In terms of U we have to solve

$$U^T\underline{x} = U^{-1}H^T r \qquad (4.5)$$

where U^T is a lower triangular matrix. The procedure of recovering data begins with the estimation of first data x_1 which is given as

$$\tilde{x}_1 = z_1/u_{11}$$

where z_1 is the first element of $\underline{z} = U^{-1}H^T\underline{r}$. x_1 is obtained by passing $\tilde{x}_1$ through the nonlinear decision device. Then x_2 is obtained as

$$\tilde{x}_2 = (z_2 - u_{21}\hat{x}_1)/u_{22}$$

We continue this procedure till $\hat{x}_m$ is obtained. Figure 4.1 gives a block schematic of the data recovery scheme based on the procedure just now outlined. In the figure $\underline{u}_i$ denotes the ith row of U^{-1} and v_i denotes ith row of U^T-D where D = diag(u_{11}, ..., u_{mm}). Although $\underline{u}_i$ and $\underline{v}_i$ are shown stored in separate register

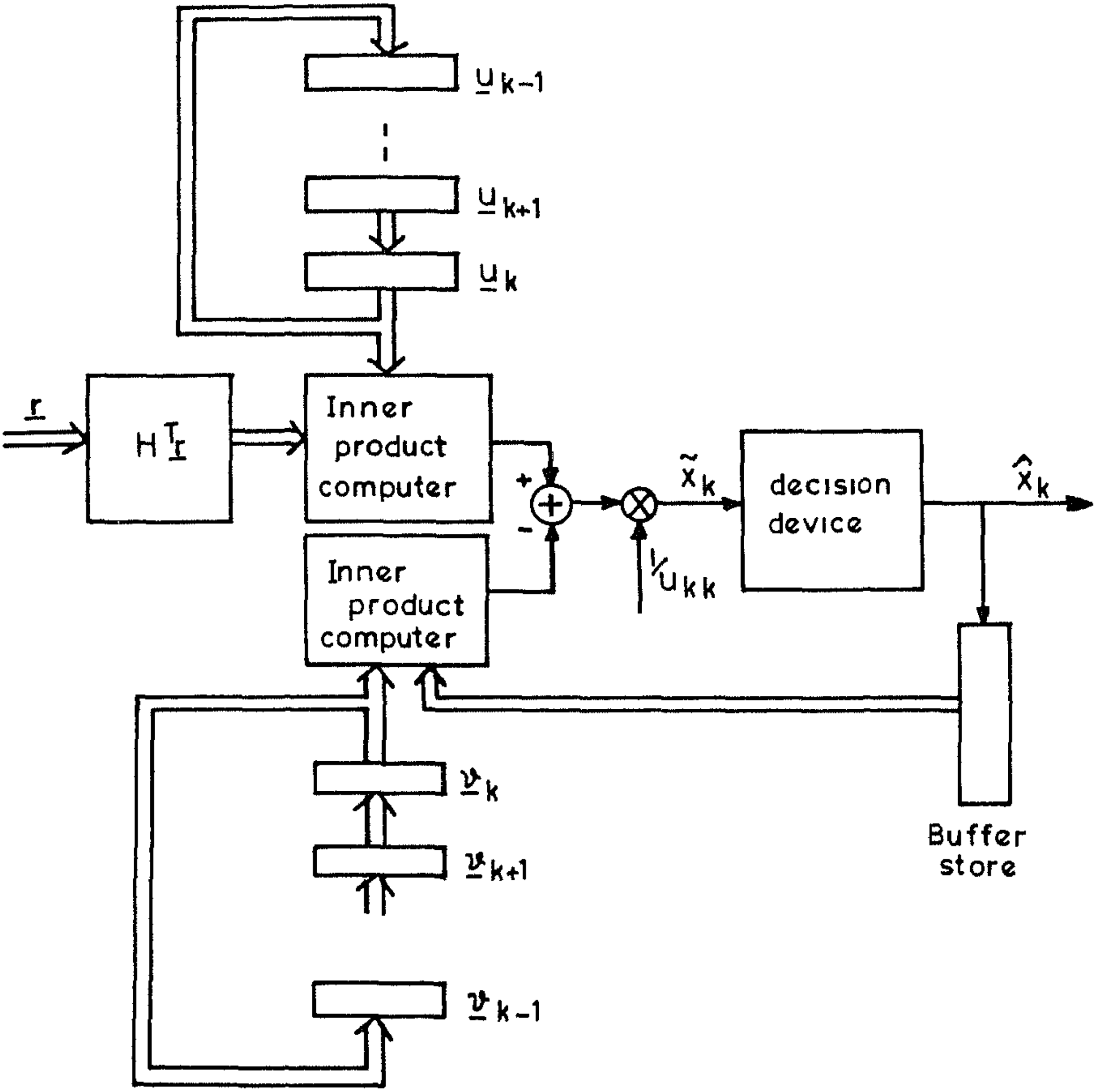


FIG 4 1 Block diagram of the data recovery scheme for BDT using Cholesky decomposition

banks in the figure in all a set of m registers of length m are sufficient

Processing of each block consists of m cycl s In the figure we have shown the status of various registers for the kth data recovery For the next data (x_{k+1}) v_{k+1} takes the place of v_k and u_{k+1} take the place of $\underline{u}_k$ u_{kk} becomes $u_{k+1\ k+1}$ At the end of m cycles the in tial state is restored and the processor is ready to d al with the next block The inner product computer shown is the major computation block in the scheme In the case of binary data th inner product computer used in the feedback loop does not require multipli cations

Computational requirements of DF DP sch m for BDT using Chole ky decomposition method is same as that of DF-DR sche e for CSDT if the block leng h is equal to the number of forward tap On th other hand the scheme shown in Fig 4 1 requires more storage compared to DF-DR schemes for CSDT syst ms How- ver with the present day technology storage requirement is not a crucial factor Whereas long bursts of error in the case of CSDT due to error propagation especially in the medium SNR range is definitely a matter of concern In the case of DF-DP scheme for BDT systems error propagation is restricted to a block length in the worst situation

analytical evaluation of the performance of DF-DR schemes is difficult due to the presence of nonlinear decision device However some idea of performance can be obtained under the assumption of error-free feedback In this case probability of ith data symbol being in error $(PER)_i$ is given by

$$(PER)_i = \int_{(SNR)_i^{1/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = Q(\sqrt{(SNR)_i}) \qquad (4\ 6)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{1}{2}y^2} dy$$

$(SNR)_i$ can be computed as follows From (4 5) we see that the effective noise vector is given by

$$w = U^{-1}H^T\underline{w}$$

Therefore SNR associated with ith data symbol is given by

$$(SNR)_i = u_{ii}^2/(||u_i||^2 \; N_o/2)$$

where $||\underline{u}_i||^2$ denotes squared length of the ith row of $U^{-1}H^T$ and $N_o/2$ is the noise variance

4 2 MMSE DECISION FEEDBACK DATA RECOVERY SCHEME

In the previous section we described one method of obtaining the parameters of DF-DR scheme using Cholesky decomposition in which we did not take noise variance into account We now derive a DR scheme based on minimum mean square

error (MMSE) criterion This scheme accounts for the noise variance

We first define several vectors and matrices which will be used in the following derivation

$\underline{b} = H^T r$

$x_k = (x_1\ x_2\ \ldots\ x_{k-1})^T$

$v_k = (v_{k1}\ v_{k2}\ \ldots\ v_{k\,k-1})^T$

$u_k = (u_{k\,k}\ \ldots\ u_{k\,m})^T$

H_k = H with first k-1 columns removed

$\underline{h}_k$ = kth column of H

$\tilde{H}_k$ = H with first k-1 columns

The objective is to obtain a set of coefficients $u_{k\,k}$ $u_{k\,k+1}$ $\ldots$ $u_{k\,m}$ v_{k1} v_{k2} $\ldots$ $v_{k\,k-1}$ such that estimate of the kth data symbol given by

$$\hat{x}_k = u_{k\,k} b_k + \ldots + u_{k\,m} b_m - v_{k\,1} x_1 - \ldots - v_{k\,k-1} x_{k-1} \quad (4\ 7)$$

is closest to the actual data x_k in the mean square sense That is

$$MSE_k = E[(\hat{x}_k - x_k)^2] \text{ is minimum}$$

Here E[ ] denotes the standard expectation operator Using the principle of orthogonality in MSE minimization [7] we obtain

$$E[e_k b_j] = 0 \quad j = k \quad k+1 \qquad m$$

and $\quad E[e_k x_i] = 0 \quad i = 1\ 2 \qquad k\ 1$

where $e_k = x_k - \hat{x}_k$ Using the fact that $b_j = h_j^T Hx + h_j^T \underline{w}$ we get

$$E[e_k b_j] = E[(\underline{u}_k^T H_k^T Hx + \underline{u}_k^T H_k^T \underline{w} - \underline{v}_k^T \underline{x}_k \quad x_k)(h_j^T H\underline{x} + \underline{h}_j^T w)]$$

$$= \underline{u}_k^T H_k^T HH^T h_j \quad \underline{v}_k^T \tilde{H}_k^T \underline{h}_j - h_1^T \underline{h}_j + \frac{N_o}{2} \underline{u}_k^T H_k^T \underline{h}_j = 0 \qquad (4\ 8)$$

$$j = k \quad k+1 \qquad m$$

Similarly $E[e_k x_i] = 0$ gives

$$\underline{u}_k^T H_k^T \underline{h}_i \quad v_{k\ i} = 0 \qquad i = 1\ 2 \qquad k-1$$

$$v_{ki} = \sum_{j=k}^{m} u_{kj}\ a_{j-i} \quad \text{where} \quad a_{j-i} = h_j^T\ h_i$$

In th matrix vector notation

$$v_k = \tilde{H}_k^T H_k u_k \qquad (4\ 9)$$

Substituting (4 9) in (4 8) and rearranging terms we obtain

$$\underline{u}_k = [H_k^T[HH^T + \frac{N_o}{2} I - \tilde{H}_k \tilde{H}_k^T] H_k]^{-1}\ H_k^T \underline{h}_k$$

$$= [H_k^T H_k + \frac{N_o}{2} I]^{-1}\ (H_k^T H_k)^{-1}\ H_k^T \underline{h}_k \qquad (4\ 10)$$

But $(H_k^T\ _k)^{-1}\ H_k^T h_k = (1\ 0\ 0 \qquad 0)^T$

Therefore $\underline{u}_k$ is obtained as the first column of

$$[H_k^T H_k + \frac{N_o}{2} I]^{-1}$$

Finally the resulting MSE associated with the kth data is given by

$$MSE_k = -E[e_k x_k] = 1 - \underline{u}_k^T H_k^T h_k = 1 - \sum_{i=k}^{m} u_{k\,i}\, h_{i\,k}$$

$$= 1 - \sum_{j=0}^{g} a_j u_{k\,j+k} \qquad (4.11)$$

The mean square error given by (4.11) reflects to a great extent the quality of the DR scheme. It can be considered as a measure of the performance for it indicates the variance of the error. If MSE is significant probability of the kth data symbol being in error is large. Thus $1/(MSE)_k$ may be considered as a measure of effective SNR associated with the kth data symbol.

Components of u_k can be computed recursively using Trench algorithm [75] because the matrix $[H_k^T H_k + \frac{N_o}{2} I]$ is a positive definite Toeplitz matrix. First we calculate u_m given by

$$u_m = u_{mm} = (H_m^T H_m + \frac{N_o}{2} I)^{-1}$$

where $H_m^T H_m = \underline{h}_m^T h_m = h_o^2 + h_1^2 + \ldots + h_g^2$

The other $\underline{u}_k$'s, $k = m-1, m-2, \ldots, 1$ can be recursively obtained by following the procedure given in [75]. Thus the computational complexity is $O(m^2)$ as the procedure for

calculating $\underline{u}_l$ $l = 1\ 2$ m is same as inverting a mxm positive def nite Treplitz matrix by Trench algorithm The implementation scheme is same as the one shown in Fig 4 1 except that $\underline{u}_1$ and $\underline{v}_1$ are given by (4 9) and (4 10) respectively

4 3 DECISION FEEDBACK SCHEME BASED ON MINIMUM PROPERTY OF FOURIER EXPANSIONS

So far in our analysis no thought has been given to the problem of illconditioning of the channel autocorrelation matrix H^TH H^TH is said to be illconditioned if the ratio of maximum and minimum eigenvalue is large [74] When H^TH is illconditioned the solution obtained by inverting H^TH is very sensitive to truncation errors and noise in the received vector It is therefore essential to consider numerically stable methods Towards this end in this section a numerically stable method of solving linear system of equations based on minimum property of Fourier expansion due to Wang [58] is adopted to derive a decision feedback data recovery scheme

Minimum Property of Fourier Expansion For any finite or infinite linearly independent sequence of elements α_1 α_2 of a Hilbert space S_H there always exists an orthonormal system β_1 β_2 such that

$$\beta_j^{\prime} = \beta_j/||\beta_j|| \qquad j = 1\ 2$$

where $\beta_1 = \alpha_1$ $\beta_{j+1} = \alpha_{j+1} - \sum_{k=1}^{j} (\alpha_{j+1}\ \beta_k)\beta_k/||\beta_k||^2$ (4 12)

$$j = 1\ 2$$

This procedure is known as Gram-Schmidt orthonormalization Here $(\alpha\ \beta)$ denote inner product of α and β

Consider an element α in S_F Then x_j s which minimize the norm

$$||\alpha - \sum_{j=1}^{n} x_j \beta_j^*||$$

are known as the Fourier coefficien s of α They are given by

$$x_j = (\alpha\ \beta_j^*)$$

This property of x_j s is known as the minimum property of Fourier expansion

Now let $\underline{z}_1$ z_2 $\underline{z}_m$ be the m column vectors of H in (4 1) Since the column of H are linearly independent orthonormalizing $\underline{z}_1$ $\underline{z}_2$ $\underline{z}_m$ we obtain $\underline{y}_1^*$ $\underline{y}_2^*$ $\underline{y}_m^*$ Then we have the following result due to Wang [58]

Theorem 4 1 Let r given in (4) be an el ment of an n-dimentional Hilb rt space $\tilde{x}_j$ s which minimize the norm $||r \sum_{j=1}^{m} x_j \underline{z}_j||$ are given by

$$\tilde{x}_m = (r\ \underline{y}_m)/||y_m||^2$$

$$\tilde{x}_{m\ j} = (\underline{r} - \sum_{k=m-j+1}^{m} \tilde{x}_k z_k\ \ y_{m-j})/||y_{m-j}||^2 \qquad (4\ 13)$$

$$j = 1\ 2 \qquad m-1$$

(Proof of this theorem is not given in [58]. Therefore for the sake of completeness we prove the theorem in the following manner.)

Proof: We know that given r and y_j's the c_j's which minimize the norm

$$||r - \sum_{j=1}^{m} c_j \underline{y}_j|| \quad \text{are given by}$$

$$c_j = (\underline{r} \cdot y_j)/||\underline{y}_j||^2$$

By substituting for $\underline{y}_j$ in terms of z_j's using (4.12) and grouping the terms common to z_j's we can obtain the recursive equations given by (4.13). To see this consider

$\sum_{j=1}^{m} c_j \underline{y}_j$ and substitute for $\underline{y}_j$ using (4.12). This gives

$$\sum_{j=1}^{m} c_j y_j = c_1 z_1 + c_2\left(z_2 - \frac{(\underline{z}_1 \cdot \underline{y}_1)\underline{z}_1}{||y_1||^2}\right) + c_3\left(\quad - \frac{(\underline{z}_3 \cdot \underline{y}_1)\underline{z}_1}{||\underline{y}_1||^2}\right.$$

$$\left. - \frac{(\underline{z}_3 \cdot \underline{y}_2)\underline{y}_2}{||\underline{y}_2||^2}\right) + $$

$$+ c_m\left(\underline{z}_m - \frac{(\underline{z}_m \cdot \underline{y}_1)z_1}{||\underline{y}_1||^2} \qquad - \frac{(\underline{z}_m \cdot \underline{y}_{m-1})y_{m-1}}{||\underline{y}_{m-1}||^2}\right)$$

From the above expression it is clear that the coefficient of z_m is c_m. Thus

$$\tilde{x}_m = c_m = (r \cdot \underline{y}_m)/||y||^2$$

The coefficient of z_{m-1} gives

$$x_{m-1} = (r - x_m z_m \quad y_{m-1})/||y_{m-1}||^2$$

The rest of the coefficien s can be similarly obt ined

It may be noted that in th method discus ed above x_m is obtained first which is used in obtaining x_{m-1} by back substitution However for the sake of uniformity we wish to obtain x_1 first This can be easily don by orthonormalizing the columns of H starting from the mth column Le y_1 y_2 y_m denote the orthogonal columns ob ain d this way Then the governing equations of the data recovery s heme are given by

$$\tilde{x}_1 = (r \quad y_1)/||y_1||^2$$

$$\tilde{x}_i = (\underline{r} - \sum_{k=1}^{i-1} \hat{x}_k \underline{h}_k \quad y_k)/||y_i||^2 \qquad (4\ 14)$$

$$i = 2\ 3 \qquad m$$

where h_k is the kth column of H $\hat{x}_i$ $i = 1\ 2$ m are the es mates of the data values obtained by pa sing $\tilde{x}_i$ $i = 1$ m through a nonlinear decision device

Figure 4 2 shows he data recovery sch me Processing of each block of data consists of m cycles lajor computational load is due to the inner product computation in every cycle

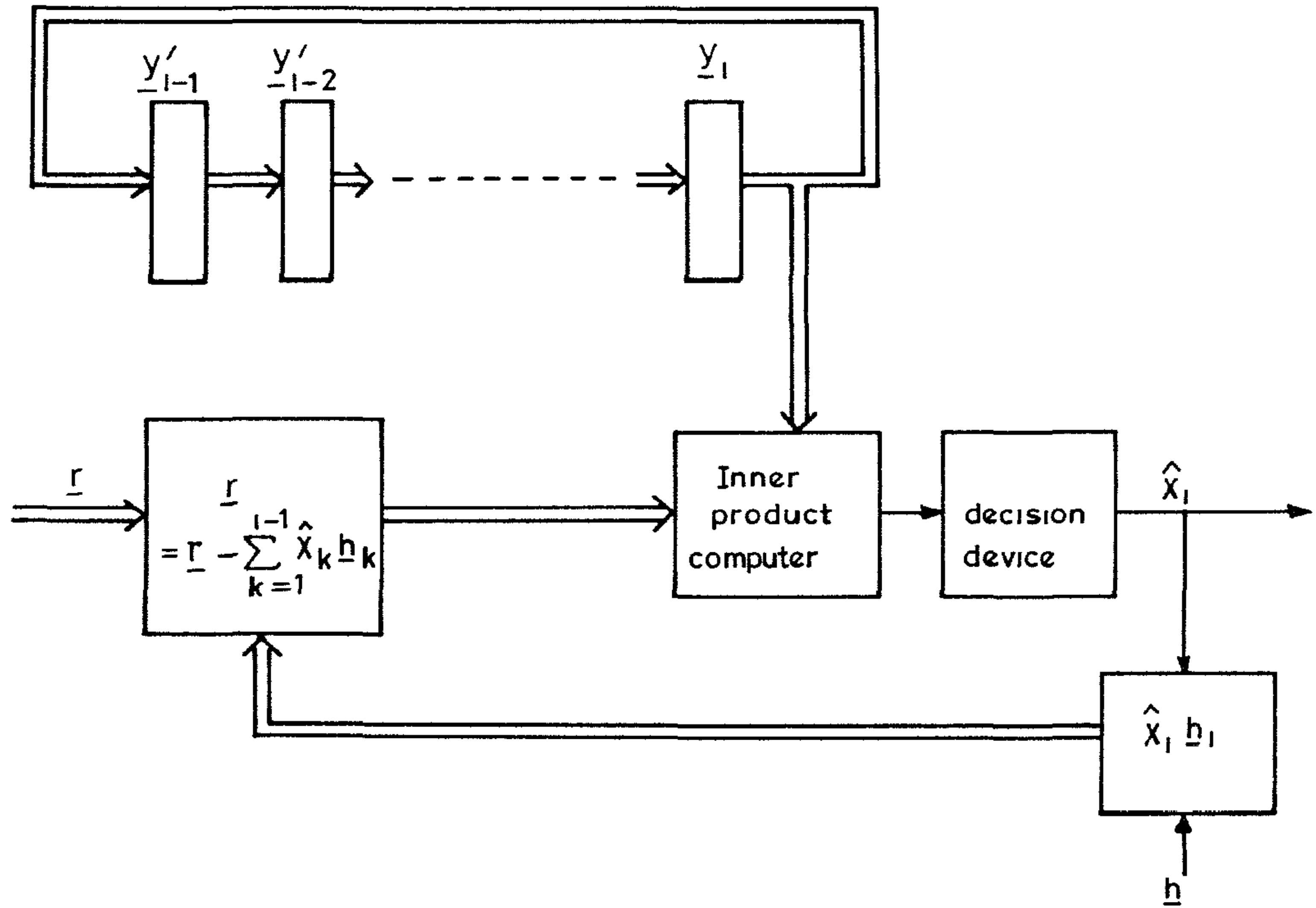


FIG 4 2 Block diagram of the data recovery scheme using minimum property of Fourier expansion

The performance of the sch me can be analytically evaluated in th absence of error The probability of error associated with ith da a symbol is given by (4 6) The SNR associat d w th the ith data symbol can be calculated as follows Substituting for r from (4 1) in (4 14) for x_i we get

$$\tilde{x}_i = (Hx + w \quad y_i)/||y_i||^2$$

$$= (\sum_{j=1}^{m} x_j(h_j \; y_i) + (w \quad y_i))/||y_i||^2$$

Substituting for h_j in t rms of y_j using (4 15) we see that

$$\tilde{x}_i = x_i + (y_i \quad w)/||y_i||^2$$

Similar expressions are obtained in the ca e of other data symbols al o Thus

$$(SNR)_i = ||y_i||^2/(N_o/2)$$

$||y_i||^2$ i = 1 2 m thus denotes a m asure of the performance of the scheme and it is related to the c annel sample r sponse vector When the columns of the channel convolutional matrix H are orthogonal $||y_i||^2 = 1$ for i = 1 2 m if $h^T h = 1$ When corr lations exist among he columns $||y_i||^2 \leq 1$ i = 1 2 m This can be seen by noting that

$$y_m = h_m$$

$$\text{and } \underline{y}_{m-i} = \underline{h}_{m-i} - \sum_{k=m-i+1}^{m} (\underline{h}_{m-i} \cdot \underline{y}_k) \underline{y}_k / ||\underline{y}_k||^2 \quad i = 1, 2, \ldots, m-1 \tag{4.15}$$

From (4.15) we obtain $||\underline{y}_m||^2 = 1$ and

$$||\underline{y}_{m-i}||^2 = 1 - \sum_{k=m-i+1}^{m} (\underline{h}_{m-i} \cdot \underline{y}_k)^2 / ||\underline{y}_k||^2 < 1$$

4.4 DECISION FEEDBACK SCHEME BASED ON MODIFIED AUSTIN'S APPROACH

One of the most widely used DR schemes in the context of CSDT system is the decision feedback equalizer. Austin [13] has derived the equations of decision feedback equalizer for CSDT systems by making suitable assumptions so that detection theory results can be easily applied. As a counterpart of this scheme, in this section we derive a DF-DR scheme by modifying Austin's assumptions. In the following discussion, we assume binary data transmission.

Let x_k denote the symbol on which decision is to be made. We modify Austin's assumptions in the following manner for the case of BDT.

1. Data symbols are independent Gaussian variables with zero-mean and unit variance for $i > k$.
2. x_i are known for $i < k$.
3. Noise components are independent Gaussian variables with zero mean and variance $N_o/2$.
4. x_k takes 1 or -1 value with equal probability.

Th l kelihood ratio for x_k is given by

$$L_k = \frac{p[\underline{r}/\underline{x}_k^p \quad x_k=1]}{p[\underline{r}/x_k^p \quad x_k=-1]}$$

wher $\underline{x}_k^p = (x_1 \; x_2 \qquad x_{k-1})^T$ Let $x_k^f = (x_{k+1} \; x_{k+2} \qquad x_m)^T$ denote the future data value with respect to x_k p( ) denotes probabili y den it/ function Now L_k can be alternatively expressed as

$$L_k = \frac{\int p(\underline{r}/x_k^p \quad x_k^f \quad x_k)\; p(\underline{x}_k^f)\; dx_k^f \,\big|_{x_k=1}}{\int p(r/x_k^p \quad \underline{x}_k^f \quad x_k)\; p(x_k^f)\; d\underline{x}_k^f \,\big|_{x_k=-1}}$$

Making use of the a sumptions made in the beginning of this s ction we can write

$$p(\underline{r}/\underline{x}_k^p \; \underline{x}_k^f \; x_k) = K_1 \exp[\sum_{i=1}^{} x_i b_i \quad \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \; a_{i-j}] \qquad (4\;16)$$

where K_1 is a constant and

$$b_i = \frac{2}{N_o} \sum_{j=0}^{g} h_j \; r_{j+i} \qquad i = 1 \; 2 \qquad m \qquad (4\;17)$$

$$a_i = \frac{2}{N_o} \sum_{j=0}^{g} h_j \; h_{j+i} \qquad i = 0 \; 1 \qquad m-1 \qquad (4\;18)$$

Dumping all the factors which cancel out while forming L_k into K_2 we rewrite (4 16) as

$$p(r/\underline{x}_k^p\ x_k^f\ x_k) = K_2\ \exp[x_l b \quad + \sum_{i=k+1}^{m} x_i b_i \quad 2 \sum_{i=1}^{k-1} \sum_{j=k+1}^{m} x_i x_j a_{i-j}$$

$$- 2x_k \sum_{i=1}^{k-1} x_i a_{i-k} - 2x_k \sum_{j=k+1}^{m} x_j a_{k-j}$$

$$\sum_{i=k+1}^{m} \sum_{j=k+1}^{m} x_i\ {}_j a_{i-j}]$$

Using assumption (1) we also have

$$p(x_k^f) = K_3\ \exp[\ \frac{1}{2} \sum_{i-k+1}^{m} \sum_{j=k+1}^{m} x_i x_j\ \delta_{ij}]$$

$$\text{where}\quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

U ing the structure of multidimensional Gaussian density function and following Aus tin we can obtain the numerator of L_k as

$$\exp[x_k b_k - 2x_k \sum_{i=1}^{k-1} {}_i a\quad k + \sum_{j=k+1}^{m} \sum_{i=k+1}^{m} (\frac{1}{2} b_j - \sum_{n-1}^{k\ 1} x_n a_{j-n}$$

$$- x_k a_{j-k}) \quad q_{ji}^k (\frac{1}{2} b_i - \sum_{n=1}^{k-1} x_n a_{i-n} - x_k a_{i-k})]$$

where q_{ji}^k are the elements of m trix $Q_k = R_k^{-1}$ The elements of R_k ar given by

$$R_{ij}^k = a_{i-j} + \frac{1}{2}\ \delta_{ij} \qquad i\ j > k \qquad (4\ 19)$$

Thus we obtain

$$L_k = \exp[2b_k - 4 \sum_{i=1}^{k-1} x_i a_{i-k} \quad 4 \sum_{n=k+1}^{m} a_{n-k} [\sum_{j=k+1}^{n} (\tfrac{1}{2} b_j - \sum_{i=1}^{k-1} a_{i-j} x_i) \, q_{jn}^k]$$

Now let $g_{kj} = - \sum_{i=k+1}^{m} q_{ij}^k \, a_{i\ k} \qquad n = 1\ 2 \qquad m\ 1 \qquad j = k+1 \qquad m$ (4 20)

and $f_{kj} = 2a_{j\ k} + 2 \sum_{i=k+1}^{m} g_{ki}\, a_{i-j} \qquad k = 2\ 3 \qquad m \qquad j = 1\ 2 \qquad k-1$ (4 21)

The decision rule in terms of g_{kj} s and f_{kj} s become

$$b_k \sum_{j=k+1}^{m} g_{kj} b_j - \sum_{i=1}^{k-1} f_{ki} x_i \underset{x_k=-1}{\overset{x_k=1}{\gtrless}} 0 \qquad (4\ 22)$$

$$k = 1\ 2 \qquad m$$

Th decision rule given by (4 22) clearly reflects the structure of the data recovery schem The structure is shown in Fig 4 3 in the figure

$b = (b_1\ b_2 \qquad b_m)^T \qquad g_k = (g_{k\ k+1} \qquad g_{k\ n})^T$

$f_k = (f_{k1}\ f_{k2} \qquad f_{k\ m-1}) \qquad b_k = (b_{k+1}\ b_{k+2} \qquad b_m)^T$

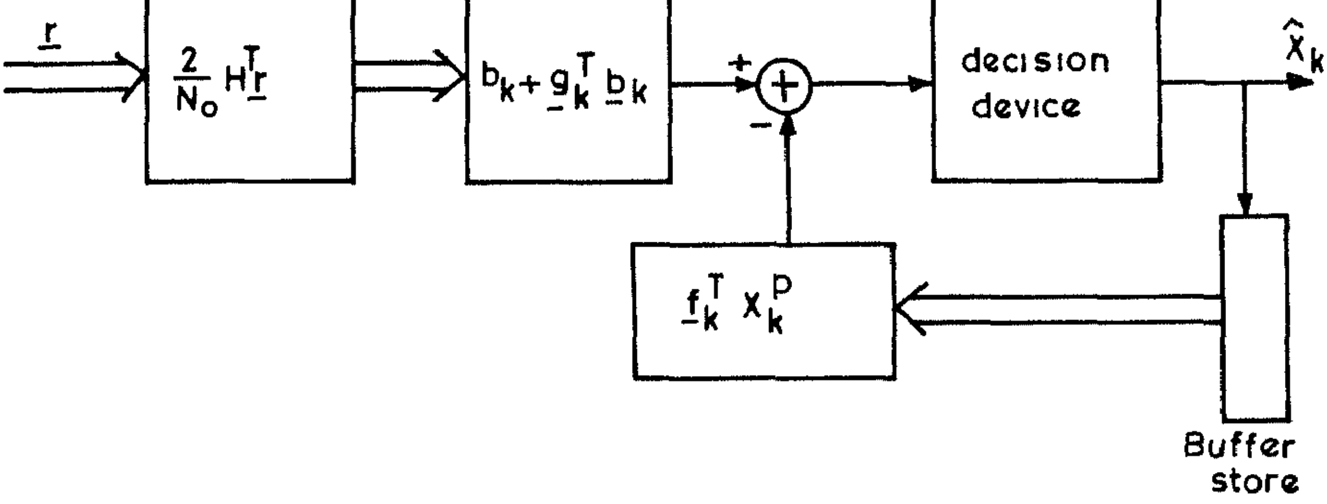


FIG 4 3 Block diagram of the data recovery scheme using modified Austins approach

In order to ıake dec s on regardıng the kth d ta symbol w need the nowledge of g_{lj} $j = k+1$ n and f_{kj} $j = 1\ 2$ $k-1$

We ıllustrate th for goıng dıscu sıon wıth an example

Example 4 1 Consıder a ock contaıııng thr data values $x = (x_1\ x_2\ x_3)^T$ transmıtt d over a chann l havıng $(h_o\ h_1\ h_2)$ a ıts fınıte ımpulse response s quence Then or a receıved vector $r = (r_1\ r_2 \quad r_5)^T$ we have

$$a_o = \frac{1}{N_o}(h_o^2 + h_1^2 + h_2^2) \qquad b_1 = \frac{2}{N_o}(r_1h_o + r_2h_1 + r_3h_2)$$

$$a_1 = \frac{1}{N_o}(h_oh_1 + h_1h_2) \qquad b_2 = \frac{2}{N_o}(r_2h_o + r_3h_1 \quad r_4h_2)$$

$$a_2 = \frac{1}{N_o}(h_oh_2) \qquad b_3 = \frac{2}{N_o}(r_3h_o + r_4h_1 + r_5h_2)$$

The matrıces R_k $k = 1\ 2$ ar gıven b

$$R_1 = \begin{bmatrix} a_o + \frac{1}{2} & a_1 \\ a_1 & a_o + \frac{1}{2} \end{bmatrix} \qquad R_2 = [a_o \quad \tfrac{1}{2}]$$

Let

$$Q_1 = \begin{bmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{bmatrix} \quad \text{and} \quad Q_2 = q_{33}$$

Then

$g_{23} = -q_{33}a_1$ $\qquad$ $f_{21} = 2a_1+2g_{23}a_2$

$g_{12} = -(q_{22}a_1+q_{23}a_2)$ $\qquad$ $f_{31} = 2a_2$

$g_{13} = -(q_{32}a_1+q_{33}a_2)$ $\qquad$ $f_{32} = 2a_1$

The decision rul (4 22) becomes

$$b_1 + g_{12}b_2 \quad g_{13}b_3 \underset{x_1=-1}{\overset{x_1=1}{\gtrless}} 0$$

$$b_2 + g_{23}b_3 - f_{21}x_1 \underset{x_2=-1}{\overset{x_2=1}{\gtrless}} 0$$

$$b_3 \quad f_{31}x_1 - f_{32}x_2 \underset{x_3=\ 1}{\overset{x_3=1}{\gtrless}} 0$$

4 1 Co iputation of the Tap Coe ficients

The co fficients g_{kj} $j = k+1$ m c n be viewed as the coef icients of the forward fil er in he cas f x_k Moreov r these co fficient can be calculated efficiently using Levinson-Durbin algorithm [7] ior that pu pose (4 20) is r written below in the matrix vector form

$$\underline{g}_k = - Q_k\underline{a}_k$$

where $\underline{g}_k = (g_{k\ k+1} \quad g_{k\ j})$ and $\underline{a}_k = (a_1\ a_2 \quad a_{m-k})$ Since

Q_k is the inverse of P_k we can write $\underline{g}_k$ as the solution of

$$P_k g_k = -a_k$$

where R_k is defined by (4 19)

R_k is a symmetric Toeplitz matrix Therefo ve can make use of Levinson-Durbin recur ive algorithm given in [7] for obtaining $\underline{g}_k$ s The required st ps for computing g_k s are given below

1 $\quad d_{11} = -a_1/(\ _o + \frac{1}{2}) \qquad _o = a_o + \frac{1}{2}$

2 $\quad d_{ii} = (a_i - \sum_{j=1}^{i-1} d_{i-1\,j}\, a_{i-j}) / e_{i-1}$

3 $\quad d_{ij} = d_{i-1j} - d_{ii}\, d_{i-1\,i\,j} \qquad 1 < j < i-1$

$\qquad i = 2\ 3 \qquad k-1$

4 $\quad e_i = (1 \quad d^2_{ii}) \quad _{i-1}$

5 $\quad g_{n-1\ m} = -d_{11}$

6 $\quad g_{k\ k+i} = d_{n-1\ i} \qquad i = 1\ 2 \qquad m \qquad k = 1\ 2 \qquad m-1$

4 4 2 Modified Au tin s Approach and Nonlinear Gaussian limina ion l thod

In this sect on we sho that the modified Austin s approach may be vi wed as nonlinear Gaussian elimination method with appropriate odifications For hi purpose (4 22) may be written using ma rix vector no ation a

$$\begin{bmatrix} 1 & g_{12} & g_{13} & & g_{1m} \\ 0 & 1 & g_{23} & & g_{2m} \\ 0 & 0 & 1 & & g_{3m} \\ & & & & \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ b_m \end{bmatrix} - \begin{bmatrix} 0 & 0 & & 0 \\ f_{21} & 0 & & 0 \\ f_{31} & f_{32} & & 0 \\ & & & \\ f_{m1} & f_{m2} & f_{m\,m-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_m \end{bmatrix} \gtrless \underline{0}$$

$$Gb - Fx \gtrless 0 \qquad (4.23)$$

where $\underline{0}$ is a null vector and $\gtrless$ denotes symbolically the nonlinear decision device. Note that the right hand side of (4.22) is an estimate of x_k. Therefore the right hand side of (4.23) is an estimate of the data vector x. That is

$$Gb - Fx = I_m \underline{x}$$

Let $\Gamma = F + I_m$. Then the above equation can be written as

$$G^{-1}\,\Gamma\,x = b$$

(G is invertible since the diagonal elements are nonzero)

But $\underline{b} = H^T \underline{r}/(N_o/2)$ which gives

$$\frac{2}{N_o} G^{-1} \Gamma\,\underline{x} = H^T r \qquad (4.24)$$

Comparing (4.24) with (4.2) it can be seen that $\frac{2}{N_o}G^{-1}\Gamma$ plays the role of $H^T H$. Thus we may call $\frac{2}{N_o}G^{-1}\Gamma$ as effective autocorrelation matrix. With this interpretation we see that modified Gaussian approach is equivalent to nonlinear Gaussian elimination method.

4 5 SO1E OBSEPV TIONS

As we are dealing with the problem of sol/ing lin ar equa- ors in the presence of noise when the unknown variables take va u s from a finite set it is natural to think tha available methods for solving linear equations must apply with appropri te modifications We demonstrated this b considering four DF-DR sche1es which are centrally based on th dea of nonlin ar Gaussian elimination method

The two DF-DR chemes considered in Section 4 1 nd S ction 4 3 essentially modify already existing me ho for solving linear equations by incorporating d cision feedback as explained in he beg ning of the chapter However the methods discussed in S ctions 4 2 and 4 4 re derived based on a set of as umpt ons and a performance criterion DF DR scheme of S ction 4 2 employs mathem tically tractabl MMSE criterion while in Section 4 4 we used more difficult e on eory approach In this approach he complexity of the scheme is lleviated considerably by assuming that the data value are indep ndent G ussian variables The major nderlying assumption in both the sch m s is that the past decisio s are rro ree

In the absence of noise all the four sch 1es no doubt provide error free perfor1ance even withou th nonlinear decision device how far d c sion f edback helps to improve

the performance ir the pres nce of noise can only bc evaluated with the hclp of sinulation Although decision edback is an old idea [77] no useful mathematic l analysis is ava_lable which deals with analytical evaluation of performance The majo tumbling block appears o he th nonlin ar decision evice which d fies y analysis In Chapt r 8 we give simulation results on DF DF schemes d veloped in sections 4 3 and 4 4

CHAPTER 5

DATA RECOVERY SCHEMES USING k CIRCULANT COMPLETIONS

In the previous chapter we made use of past decisions and banded Toeplitz nature of the channel autocorrelation matrix H^TH for deriving decision feedback deconvolution schemes which are suboptimal compared to the ML deconvolution scheme of Chapter 3. It may be observed that so far in our study no attention has been given to the convolution matrix H of the channel. We shall see in this chapter that H has some interesting properties compared to H^TH from the point of view of reducing computations and storage requirements. Based on k-circulant completion of H we develop two data recovery schemes. These schemes are easily implementable and have reduced computational complexity. Furthermore unlike the DF DP schemes of Chapter 4 which are nonlinear, the DR schemes studied in this chapter are linear and they permit analytical evaluation of their performance.

The chapter is organized as follows. In Section 5.1 we introduce the notion of k-Circulant completion and describe its application to deconvolution. Two methods of inverting k-Circulant matrices are given in Section 5.2. The first method uses FFT for inverting k-circulant completion of H. Complexity

of his algorithm is $O(n\log_2 n)$ wh re n is the number of rows of H Th second method makes use of n rsity nd block k-Circulant nature of the k-Circulan compl tion of H to provid an $O(n\log_2 g_1)$ algorithm where $g_1 = g\ 1$ In S ction 5 3 we discu s two DP schc n s which are based o k-DFT domain d convolution discussed in Scc ion 5 1 and direct inversion of c Circulant completion of H d scribed in c ion 5 2 In Section 5 4 w obtain explicit e pression for probability o rror and compare it with the performance r DF scheme based on least quare d convolution metl od

5 1 K-CIRCULANT COMPLLTION AND FINITE DECONVOLUTION

Consider the received vector $\underline{r}$ given by (2 7)

$$\underline{r} = Hx + w \tag{5 1}$$

When no statistical characte ization of $\underline{x}$ and w are available we know that the maximum likelihood estimat of x under the assumption of r al valued data is giv n by

$$x_L = (H^T H)^{-1} H^T \underline{r}$$

The most time con uming op ration in obt ining x_{LS} is the inv r ion of $H^T H$ Ev n ir we assume that $(H^T H)^{-1} H^T$ is computed before the transmission of data memory requirements can be come prohibitive Therefore even with the availabili y of s veral fast algorithms for inverting $H^T H$ [78] [80] there is

still need or processing algori hms th t are easily implementable In this section we consider the applic tion of k Circulant matrices [81] [82] for olving deconvolu on problem or he sake of e sy reference so e of the results r garding k-circulant ma rices are given in ppendix D

As an illustration we consider (5 1) and it 1-Circulant count rpart for n = 5 m = 3 and g = 2 In this case (5 1) can be written as

$$\begin{bmatrix} r_o \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} h_o & 0 & 0 \\ h_1 & h_o & 0 \\ h_2 & h_1 & h_o \\ 0 & h_2 & h_1 \\ 0 & 0 & h_2 \end{bmatrix} \begin{bmatrix} x_o \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_o \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} h_o & 0 & 0 & kh_2 & kh_1 \\ h_1 & h_o & 0 & 0 & kh_2 \\ h_2 & h_1 & h_o & 0 & 0 \\ 0 & h_2 & h_1 & h_o & 0 \\ 0 & 0 & h_2 & h_1 & h_o \end{bmatrix} \begin{bmatrix} x_o \\ x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_o \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$\underline{r} = H_1 \underline{x}_e + w \qquad (5\ 2)$$

where H_k i the k-Circulant completion of H and x_e is obtained by padding x with two zeros It i not di icult to see that the r presentation giv n in (5 2) in terms of H_k is also valid in the general case when r is fini e (n = m+g) and x is padded with n-m zeros to obtain x_e

From now on we call H_1 the Circulant completion of H as it is easy to se from (5 2) that H_1 is a Circulant matrix Similarly H_{-1} will denote the Skew Circulant completion of H

5 1 1 Deconvolution Usin Circulant Completion

Consider the circulant completion of H If R X and W denote DFT of r x and w respectively using the property of cyclic convolution [59] we can write (5 2) as

$$\underline{R} = H X + \underline{W} \qquad (5\ 3)$$

where $\underline{H}$ denote the DFT of the first column of H (H $\underline{X}$) denotes point wise multiplication of the components of the individual vectors Equation (5 3) suggests an easy way of deconvolving whenever none of the component of $\underline{H}$ is too small or zero In such cases an estimate of x can be obtained as

$$\underline{x}_{1e} = F^{-1}(\underline{R}/H) \qquad (5\ 4)$$

where F denotes the DFT matrix of order n and $(\underline{R}/\underline{H})$ is a vector obtained by componentwise division This method of deconvolution can be attributed to Cooley [83] who after becoming instrumental in the development of FFT algorithm [84] proposed the foregoing approach as an application of FFT in the context of signal processing Some of the problems associated with such straightforward application to situations involving erroneous measurements have been discussed at length by Hunt[59]

5 1 2 Deconvolution Using k-Circulant Completion

In this section we show that (5 3) and (5 4) can also be generalized to the case of k-Circulant completions For this

purpose let $\underline{b} = (b_0\ b_1\ \ldots\ b_{n-1})^T$ and $c = (c_1\ c_1\ \ldots\ c_{n-1})^T$ be two vec ors We define k-cyc ic convolution of b and $\underline{c}$ as anothe vector d given by

$$d_i = \sum_{j=o}^{i} b_j\ c_{i-j} + k \sum_{j=i+1}^{n-1} b_j\ c_{(i-j)\mathrm{mod}n} \qquad (5\ 5)$$

We define k-DFT of $\underline{b}$ and c as (see Appendix B)

$$B_{ki} = \sum_{j=o}^{n-1} b_j(\lambda w^i)^j \qquad i = 0\ 1\ \ldots\ n-1 \qquad (5\ 6)$$

$$C_{ki} = \sum_{j=o}^{n-1} c_j(\lambda w^i)^j \qquad i = 0\ 1\ \ldots\ n-1 \qquad (5\ 7)$$

respectively The corresponding inverse tran forms are

$$b_i = \frac{1}{r} \sum_{j=o}^{n-1} B_{kj}(\lambda w^j)^i \qquad i = 0\ 1\ \ldots\ n-1 \qquad (5\ 8)$$

$$c_i = \frac{1}{n} \sum_{j=o}^{n-1} C_{kj}(\lambda w^j)^i \qquad i = 0\ 1\ \ldots\ n-1 \qquad (5\ 9)$$

where λ is an nth root of k and $w = e^{j2\pi/n}$ Now we prove the following theo em

Theorem 5 1 If B_l and C_k represent k-DFT of b and $\underline{c}$ resp cti ly giv n by (5 6) and (5 7)

$$d_i = \frac{1}{n} \sum_{j=o}^{n-1} B_{kj}\ C_{kj}\ (\lambda w^j)^{-i} \qquad i - 0\ 1\ \ldots\ n-1 \qquad (5\ 10)$$

or $\qquad \underline{D}_k = (B_k\ C_k) \qquad (5\ 11)$

where D_k is the k-DFT of d

Proo Substituting for b_j and c_j from (5 8) and (5 9) in (5 5) we obtain

$$d_i = \sum_{j=0}^{i} \frac{1}{n} \sum_{t=0}^{n-1} B_{kt}(\lambda w^t)^{-j} \frac{1}{n} \sum_{s=0}^{n-1} C_{ks}(\lambda w^s)^{(i-j)}$$

$$k \sum_{j=i+1}^{n-1} \frac{1}{n} \sum_{t=0}^{n-1} B_{kt}(\lambda w^t)^{-j} \frac{1}{n} \sum_{s=0}^{n-1} C_{ks}(\lambda w^s)^{-(i-j+n)}$$

Sinc $\lambda^n = k$ on r arranging the above expres ion becomes

$$d_i = \frac{1}{n^2} \sum_{j=0}^{i} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} B_{kt} C_{ks} \lambda^{-i} w^{-s} w^{j(s-t)}$$

$$+ \frac{1}{n^2} \sum_{j=i+1}^{n-1} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} B_{kt} C_{ks} \lambda^{-i} w^{si} w^{j(s-t)}$$

Ch nging the order of summation it becomes

$$d_i = \frac{1}{n^2} \sum_{=0}^{n-1} \sum_{s=0}^{n-1} B_{kt} C_{ks} \sum_{j=0}^{n-1} \lambda^{-i} w^{-si} w^{j(s-t)}$$

$$\text{But} \quad \sum_{j=0}^{n-1} w^{j(s-t)} = \begin{cases} n & \text{if} \quad s=t \\ 0 & \text{if} \quad s\neq t \end{cases}$$

Th refore $d_i = \frac{1}{n} \sum_{s=0}^{n-1} B_{ks} C_{ks}(\lambda w^s)^{-i} \quad i = 0\ 1 \qquad n-1$

*

Using (5 11) we easily ee that if $\underline{H}_k$ R_k X and W_k represent k-DFT of h $\underline{r}$ $\underline{x}$ and w respectively then

$$R_k = \underline{H}_k X_k + W_k$$

From which the deconvolution solution can be obtained as

$$\underline{x}_k = F_k^{-1}\ (R_k/\underline{H}_k) \qquad (5\ 12)$$

provided none of the component of $\underline{H}_k$ are zero

5 1 3 Need for Deconvolution by k Circulant Completion

Note that when some of the DFT coefficients of the channel are zero (4) cannot be used for deconvolution as division by zero is not permitted In such cases even if H_1 is singular for some other k H_k may be invertible (i all the k-DFT coefficients are nonzero) Thus in some cases it may be more advantageous to use k DFT instead of 1-DFT As an illustration consider a simple example of a channel having the following H matrix when n = 4 and m = 3

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad H_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad H_{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

It is not difficult to see that H_1 (Circulant completion) is singular whereas H_{-1} the Skew Circulant completion of H is nonsingular In fact $\det(H_k)$ for this channel is 1-k Therefore H_k is nonsingular for $k \neq 1$

It may be further noted that computation of k-DFT is as simple as 1-DFT computation by FFT To see this consider

k DFT of $\underline{b}$ j ven by (5 6) It can be alternatively wr_tten as

$$B_{ki} = \sum_{j=o}^{n\ 1} (b_j \lambda^j)\ w^{ij}$$

where λ is an nth root of k Computation of B_{ki} s may be arried out by comp uting FFT of $b_j \lambda^j$ s j = 0 1 n-1 instead of b_j s Th s aspect ma es -DIT application more attractive

5 2 DECONVOLUTION BY DIR CT I VERSIO

Th stimate x_{1e} or x_k obta ned via 1 DFT and k-DFT given by (5 4) and (5 12) can als be obtain d by directly inverting H_k whenev r H_k is nonsingular Th refore we can rite

$$\underline{x}_{ke} = H_k^{-1}\ r \tag{5 13}$$

we now pre ent two methods of inverting H_k

Me hod 1

H_k can be expressed as (see Appendix B)

$$H_k = D^{-1}\ C\ D$$

where D is a diagon l matrix with 1 λ λ^2 λ^{n-1} as the diagonal ntries C is a Circulant matrix with first column elements $D\underline{h}$ where h is the c nn l impulse response vector Therefore C^{-1} can b obtain d by using FT Then H_k^{-1} is given by

$$H_k^{-1} = D^{-1} C^{-1} D$$

when n is a power of 2 inversion of C can be carried out using $2n\log_2 n + n$ complex arithmetic operations [82] Therefore the complexity of this method is $O(n\log_2 n)$

Elements of the first column of H_k^{-1} are obtained as follows Let $C_0\ C_1 \ldots C_{n-1}$ denote first column elements of C^{-1} If $C_0\ C_1 \ldots C_{n-1}$ denotes the first column elements of H_k^{-1} then C_i are given by $C_i = C_i/\lambda^i$ $i = 0\ 1 \ldots n-1$

Method 2

In this method sparse nature of H_k is exploited to derive an $O(n\log_2 g_1)$ complexity algorithm for inverting H_k^{-1} First we illustrate the method for Circulant matrices when $n = 2g_1$ where $g_1 = g+1$ Let $n = 8$ and $g = 3$ Then the Circulant matrix has the following form

$$H_1 = \begin{bmatrix} h_o & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 \\ h_1 & h_o & 0 & 0 & 0 & 0 & h_3 & h_2 \\ h_2 & h_1 & h_o & 0 & 0 & 0 & 0 & h_3 \\ h_3 & h_2 & h_1 & h_o & 0 & 0 & 0 & 0 \\ 0 & h_3 & h_2 & h_1 & h_o & 0 & 0 & 0 \\ 0 & 0 & h_3 & h_2 & h_1 & h_o & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_2 & h_1 & h_o & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_o \end{bmatrix} = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$$

Note that h_1 is a sparse matrix Direct application of FFT means computation of DFT of the first column having 4 zeros Another more efficient approach is to write H_1^{-1} as

$$H_1^{-1} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Whenever H_1^{-1} exists $H_1^{-1}H_1 = I_n$ Thus we obtain

$$\begin{aligned} PA + QB &= I_{g_1} \\ PB + QA &= O_{g_1} \end{aligned} \qquad (5\ 14)$$

Solving the two equations we obtain

$$A+B = (P+Q)^{-1} \quad \text{and} \quad A-B = (P-Q)^{-1}$$

where

$$P+Q = \begin{bmatrix} h_o & h_3 & h_2 & h_1 \\ h_1 & h_o & h_3 & h_2 \\ h_2 & h_1 & h_o & h_3 \\ h_3 & h_2 & h_1 & h_o \end{bmatrix} \qquad P-Q = \begin{bmatrix} h_o & -h_3 & -h_2 & -h_1 \\ h_1 & h_o & h_3 & -h_2 \\ h_2 & h_1 & h_o & -h_3 \\ h_3 & h_2 & h_1 & h_o \end{bmatrix}$$

Clearly P+Q is a circulant while P-Q is a Skew Circulant Note that both are full matrices (no zero entries) If g_1 is a power of 2 we can invert (P+Q) and (P-Q) using $O(n\log_2 g_1)$ computations

Now let us consider the general case where g_1 divides n such that n = rg_1 $(r > 2)$. We can partition H_1 in this case as

$$H_1 = \begin{bmatrix} A_o & 0 & 0 & & 0 & A_1 \\ A_1 & A_o & 0 & & & 0 \\ 0 & A_1 & A_o & & & 0 \\ 0 & & & & & 0 \\ & & & & & \\ 0 & & & & A_1 & A_o \end{bmatrix}$$

Let H_1^{-1} exist and

$$H_1^{-1} = \begin{bmatrix} B_o & B_{r-1} & & B_1 \\ B_1 & B_o & & B_2 \\ & & & \\ B_{r-1} & B_{r-2} & & B_o \end{bmatrix}$$

Using the identity $H_1^{-1}H_1 = I_n$ we obtain

$$\begin{aligned} A_oB_o + A_1B_{r-1} &= I_{g_1} \\ A_oB_{r-1}+A_1B_{r-2} &= 0_{g_1} \\ A_oB_{r-2}+A_1B_{r-3} &= 0_{g_1} \\ & \\ A_oB_1 + A_1B_o &= 0_{g_1} \end{aligned} \qquad (5.15)$$

By adding the abov equations we obtain

$$(B_o + B_1 + \quad B_{r-1}) = (A_o + \quad {}_1)^{-1} \qquad (5\ 16)$$

Now consid r rth roots of unity They are

$$w^i \quad i = 0\ 1 \qquad r-1 \quad \text{where} \quad w = e^{j2\ /i}$$

Multipying ith equation in (5 1) $0 < i < r-1$ by w^i and adding we can obtain

$$(B_o + w^{r-1}B_1 + \qquad wB_{r-1}) = (A_o + w^{r-1}A_1)^{-1} \qquad (5\ 17)$$

Similarly multiplying ith equation in (5 14) by w^{ik} and adding we get the general rel tionship

$$\sum_{i=o}^{r-1} B_i w^{ik} = (A_o + w^{r-k}A_{r-1})^{-1} \quad k = 0\ 1 \qquad r-1 \qquad (5\ 18)$$

which contains (5 16) and (5 17) as particular cases

Thus in order to determine B_i s we have to form r k Circulants (k will be different for each Circulant) whose inver ion can be achieved using $O(n\log_2 g_1)$ computations when g_1 is a power of 2 Finally B_i s are obtain d using (5 18) with $O(g_1\log_2 g_1)$ more computations Thus the to al complexity of the method is $O(n\log_2 g_1)$ whi h grows linearly with n for fixed g_1

Extension to the ca e of H_k^{-1} is straightforward In this cas for $n = rg_1$ we can write

$$H_k = \begin{bmatrix} A_o & 0 & 0 & & 0 & kA_1 \\ A_1 & A_o & 0 & & 0 & 0 \\ 0 & A_1 & A_o & & & 0 \\ & & & & & \\ 0 & 0 & & & A_1 & A_o \end{bmatrix}$$

Since inverse of a k-C rculant is also a k-Circulant w can partition H_k^{-1} as

$$H_k^{-1} = \begin{bmatrix} B_o & kB_{r-1} & & kB_1 \\ B_1 & B_o & & kB_2 \\ & & & \\ B_{r-1} & B_{r-2} & & B_o \end{bmatrix}$$

Thus equation (5 14) becomes

$$\begin{aligned} A_oB_o + kA_1\ B_{r-1} &= I_{g_1} \\ A_oB_{r-1} + A_1\ B_{r-2} &= 0_{g_1} \\ & \\ A_oB_1 + \ _1B_o &= 0_{g_1} \end{aligned} \qquad (5\ 19)$$

Now let λ be an rth root of k Then multiplying ith equation in (5 19) $0 \le i \le r-1$ by $(\lambda w)^{ij}$ and adding we obtain

$$\sum_{i=0}^{r-1} B_i w^{ij} \lambda^i = ({}_0F(\lambda w)^{r-j} A_i)^i \quad j = 0\ 1\ 2 \quad r-1 \tag{5 20}$$

It may be noted that he p ocedur is s mila to the case of Circulant matri inversion

5 3 DATA RECOVERY SCHEMES BASED ON k-CIRCULANT COMPLETIONS

When SNR is high the k-DFT pproach describ d in Section 5 1 2 can be used for r cov ring data Such a scheme is shown in Figure 5 1 Although two FFT proce sors are shown in the figure one is sufficient in actual practice for at a time only one of them will be in use

Note that in the case of the DR scheme shown in Fig 5 1 for every block of data transmitted k-DFT of the received vector and an inverse DFT i to be performed One way of avoiding calculation of DFT for every block separ tely is to use the elements H_k^{-1} directly in the DR schem Such a DR sch m is shown in Fig 5 2 This scheme involves a circulating hift register of length n in which fir t row elements of H_k^{-1} are stored and circulated The ta coeffici nts of the cyclic transversal filter are the received observations r_0 r_1 r_{n-1} The multiplier with coefficient k in the feedback loop of the circulating filter dep nds upon the completion used

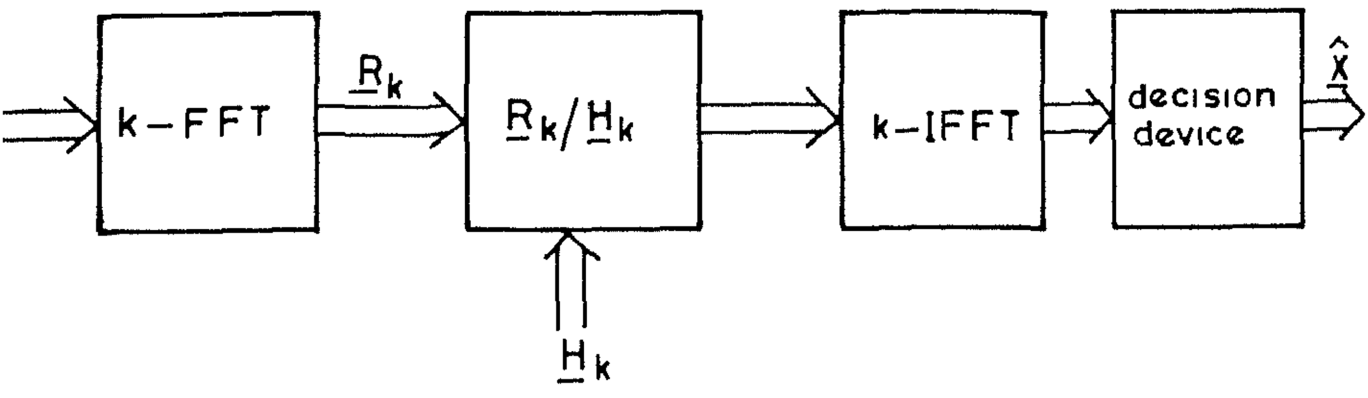


FIG 5 1 Block diagram of the data recovery scheme using k-DFT

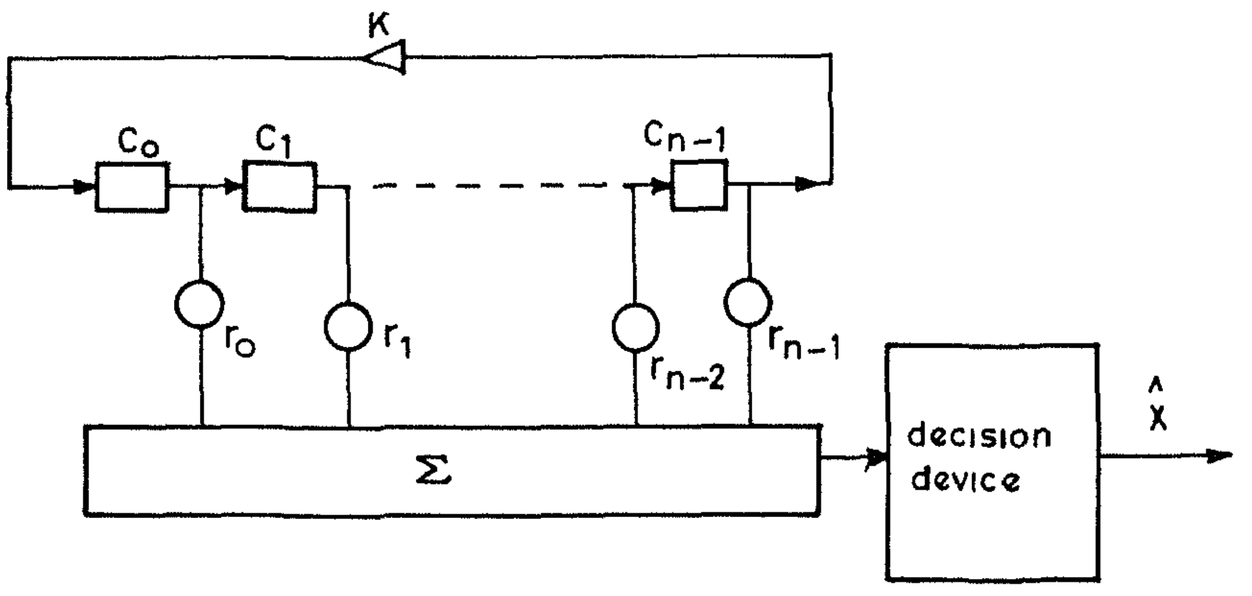


FIG 5 2 Data recovery scheme based on k-Circulant completion

Processing of each data block consists of m cycles If the entire processing is completed in nT seconds where T is the symbol duration there will be no need for long buffers to store the incoming blocks In this case the delay involved will be one block duration Thus two registers of length n are sufficient One is used in the processing while the other one stores observations arriving due to the next block

The complexity of the DP scheme shown in Fig 5 1 depends on the method used for implementing FFT Since FFT algorithm is parallel and recursive it is suitable for those real time applications where cost of the DR scheme is more important than the probability of error performance

major advantage of the DR scheme shown in Fig 5 2 is that in its implementation complex arithmetic is totally avoided Moreover its implementation is simple compared to DR schemes considered in the previous chapters

5 4 PERFORMANCE CONSIDERATIONS

In the context of data transmission it has generally been observed that the complexity of DR schemes seem to be directly proportional to the desired performance level Thus minimum probability of error (the best performance) demands highly complex DR schemes Therefore we may infer that DR schemes based on k-Circulant completion being less complex

their performance may not be satisfactory compared to DP schemes of chapters 3 and 4 However the DP schemes considered in this chapter have a distinct advantage over DR schemes developed in the previous chapters when the SNR is high In this section we obtain explicit expressions for the probability of error and compare it with the performance of DR scheme based on least square deconvolution Simulation studies on the performance of the DP schemes studied in this chapter are presented in Chapter 8

To begin with let k = 1 Substituting for r from (5 1) in (5 1) we obtain for k = 1

$$\tilde{x}_{1e} = \underline{x}_e + H_1^{-1}\,\underline{w} \qquad (5\ 21)$$

From (5 21) we see that the effective noise variance depends on the elements of H_1^{-1} Since the elements in any row is a cyclically shifted version of the first row and since noise components are statistically independent and are of equal variances the effective variance σ_1^2 is given by $\sigma_1^2 = \sigma_2 d_1^2$ where

$$d_1^2 = \sum_{i=1}^{n} h_{1i}^2 \qquad (5\ 22)$$

and $\underline{h}_1$ is the first column of H_1^{-1} With the knowledge of the DFT coefficients of $\underline{h}$ we can express d_1^2 alternatively as

$$d_1^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\lambda_i \lambda_{n-i+2}}\right) \qquad \lambda_{n+1} = \lambda_1 \qquad (5\ 23)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the components of $\underline{H}$ or the eigenvalues of H_1. The following theorem provides a lower bound on d_1^2.

Theorem 5.2 : d_1^2 as defined by (5.22) is lower bounded by

$$d_1^2 \geq \frac{1}{n(\sum_{i=1}^{g} h_i^2)} \tag{5.24}$$

Proof :

We know that for a nonsingular matrix A of order n

$$AA^{-1} = I_n$$

which implies $||AA^{-1}||_e = ||I||_e$ or $||I||_e \leq ||A||_e \; ||A^{-1}||_e$

$$\tag{5.25}$$

where $||A||_e$ denotes the Euclidean norm of the matrix A given by

$$||A||_e = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2)^{1/2}$$

Now let $A = H_1$; then $||H_1||^2 = n \sum_{i=1}^{g} h_i^2$ (5.26)

If h_{1i} $i = 1,2, \ldots, n$ denote the elements in the first column of H_1^{-1} we have

$$||H_1^{-1}||_e^2 = n \sum_{i=1}^{n} h_{1i}^2 \tag{5.27}$$

Using (5 25) (5 26) and (5 27) v obtain

$$n < n^2(\sum_{i=1}^{n} h_{1i}^2)(\sum_{i=1}^{g} h_i^2) \quad \text{or}$$

$$\sum_{i=1}^{n} h_{1i}^2 > \frac{1}{n(\sum_{i=1}^{g} h_i^2)}$$

Using (5 22) in the above inequality we obtain (5 24)

(5 22) and (5 24) are also valid in the case of k Circulant completion since $|k| = 1$ Therefore we can write

$$d_k^2 = \sum_{i=1}^{n} |h_{Ki}|^2 > \frac{1}{n(\sum_{i=1}^{g} h_i^2)} \tag{5 28}$$

It is interesting to note that all the data symbols recovered have the same probability o being in error because SNR associated with each of them is same It is given by

$$(\text{SNP})_i = \frac{1}{d_k^2 \frac{N_o}{2}} \qquad i = 0\ 1 \qquad m-1 \tag{5 29}$$

When $|k| = 1$ The probability of error (PER) is given by

$$\text{PER} = \int_{\frac{1}{d_k \sqrt{(N_o/2)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy = Q\left(\frac{1}{d_k \sqrt{(N_o/2)}}\right) \tag{5 30}$$

Let us now consider th probability of rror performance of the DR scheme based on LS-deconvolution solution given by (5 2) In this case SNR associated with data symbols will be different The average probability o error is given by

$$\overline{PEP}_{LS} = \frac{1}{m} \sum_{i=1}^{m} Q\left(\sqrt{\frac{2}{d_{Li}^2 N_o}}\right) \tag{5 31}$$

where $d_{Li}^2 \frac{N_o}{2}$ is the variance of the noise componen associated with ith data symbol An pper bound for PER_{LS} can be obtained as

$$PER_{LS} < Q\left(\sqrt{\frac{2}{d_{max}^2 N_o}}\right) \tag{5 32}$$

where $d_{max}^2 = \max_{1 \leq i \leq m} d_{Li}^2$

We note that d_{Li}^2 is the squared sum of the elements of the ith row of $(H^T H)^{-1} H^T$ Equivalently they are the diagonal elements of $(H^T H)^{-1}$ (because $(H^T H)^{-1} H^T H (H^T H)^{-1} = (H^T H)^{-1}$) We now show that

$$d_k^2 \geq d_{Li}^2 \qquad i = 1\ 2 \qquad m \tag{5 33}$$

as a particular case of a more general result Let $H_g = [H\ H_c]$ where Hg is the square completion of H such that H_g is invertible (Such completions are treated in greater detail in Chapter 6)

Let d_{gi}^2 $i = 1, 2, \ldots, n$ be the diagonal elements of $(H_g^T H_g)^{-1}$. In the following theorem we prov nat

$$d_{gi}^2 \geq d_{Li}^2 \tag{5.34}$$

It is easy to see that (5.33) follows as a special case of (5.34) when $|k| = 1$

Theorem 5.3 Given H_g, d_{gi}, d_{Li} as defined above

$$d_{gi}^2 \geq d_{Li}^2 \qquad i = 1, 2, \ldots, m$$

Proof $(H_g^T H_g)^{-1}$ can be written as

$$(H_g^T H_g)^{-1} = \begin{bmatrix} H^T H & H^T H_c \\ H_c^T H & H_c^T H_c \end{bmatrix}^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Since

$$\begin{bmatrix} H^T H & H^T H_c \\ H_c^T H & H_c^T H_c \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}$$

We have

$$H^T H\, G_{11} + H^T H_c G_{21} = I_m \tag{5.35}$$

$$H^T H\, G_{12} + H^T H_c G_{22} = 0_{m\times n-m} \tag{5.36}$$

Therefore from (5.36) w obtain

$$H^T H_c = -H^T H G_{21} G_{22}^{-1}$$

where G_{22}^{-1} exists becau e $H_g^T H_g$ is a positive definite matrix substituting for $H^T H_c$ in (35) w get

$$(H^T H)^{-1} = G_{11} - G_{12} G_{22}^{-1} G_{21} \qquad (5\ 37)$$

$G_{21} = G_{12}^T$ a $H_g^T H_g$ is a symm tric matrix

Now let $\tilde{d}_i^2$ be the diagonal el ments or $G_{12} G_{22}^{-1} G_{21}$

Since $G_{21} = G_{12}^T$ and G_{22} is a p d m tri

$G_{12} G_{22}^{-1} G_{21} > 0$ (by the definition or a p d matrix A x^T $\underline{x} > 0$ for all $\underline{x} \in R^n$ xcept $x = o$

which shows that $\tilde{d}_i^2$ is strictly positive Now using (5 37)

$$d_{Li}^2 = d_{gi}^2 \quad \tilde{d}_i^2$$

where d_{gi}^2 are the diagonal entries of G_{11}

**

Summarizing in this chapter we have s udied the application of k Circulant completions for data reco r/ in the context of BDT We have analysed two DR schemes which facilitate good reduction in omputational complex t/ and torage requirements compared to the DR schemes studied in the previous chapter Moreover these schemes are am nable to performance analysis W have obtained explicit expressions for the probability of rror and compar d it with th p rformanc achievabl by

DR scheme based on LS-decorvolution In general the performance of DR schemes based on k Circulant completions is inferior to the DR schemes of chapters 3 and 4 However in Chapter 8 we shall see that in the case of some channels DR schemes based on k-Circulant compl tion perform well compared to ML-DR and DF-DP schemes

In the next chapter w take up a detailed study of the relationship between least squares deconvolution and deconvolution via k-Circulant completions

$$\text{But } (H^TH)^{-1} H^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\text{Ther fore } \underset{\sim}{x}_{LS} = \begin{bmatrix} 1 & 1ᴗ \\ -1 & 23 \end{bmatrix} \neq \begin{bmatrix} 1 & 6 \\ -1 & 1 \end{bmatrix} = x_1$$

Thus ıt becomes ımpo tant and ınteresting to know condi- tions under wh cl $x_{L\circ} = {}_k$ b caus ın su r a case LS solu- tion can be obtaine l wıtı nuch less computation via k Circulant completions

In the present chapt r we study this problem in a more g n ral framework of square completion of rectangular matrices Let $H_s = [H \quad H_{s1}]$ be nonsingular square completion of H ın gen ral H_{s1} need not depend on H The k Circulant comple- tion of H is a particular case of H_s wh re H_{s1} depends on H Let $H_s^{-1} = \begin{bmatrix} H_{s2} \\ -H_{s3} \end{bmatrix}$ Our interest is to obtain conditions under- which

$$x_{LS} = x_s = H_{s2} \underset{\sim}{r} \qquad (6\ 3)$$

Ve how that $\underset{\sim}{x}_{L\circ} = \underset{\sim}{x}_s$ if and only if $H^TH_{s1} = O_{mxn-m}$ Such completions are called or hogonal co pletions As a corro- llary it follows that $x_L = \underset{\sim}{x}_k$ iff $H^TH_{k1} = O_{mxn-m}$ Under this condition we fur h r show that orthogonal k-Circulant completions of H are orthogonal k Circul nt matrices

We organize the chapter in the following manner In Section 6 1 a method of solving overdeterm in d set of linear equations using the notion of squar completion of matrices is discussed A brief review of least squares solution of lin ar system of equations and Moor -Penrose (MP) inverse of matric s is given in Section 6 2 In Section 6 3 we obtain conditions on squar completions that result in P-inv rse or least squares inverse (LS-inverse) of a full ank rectangular matrix Squar completions which result in MP-inverse are called ortho gonal square completions W further show that H_k is an orthogonal k Circulant completion of H iff H_k is an orthogonal k-Circulant In Section 6 4 we determine channels which are amenable to LS solution via k-Circulant completions We show that in most of the practic l cases LS solution to the deconvolution problem cannot be obta ned via (6 2) because in such cases k-Circulant completions do not result in orthogonal k-Circulants In Section 6 5 we characteriz orthogonal k-Circulants in the k-DFT domain An alternat expre sion for LS solution using partitions of k-Circulant completions of H is obtained in Section 6 6 for the purpo e of knowing the relationship between H_{k2} and th MP-inver of H We show that

$$H^{-} = H_{k2} - H_{k2} F_k \; (H_{k3} H_{k3}^{*})^{-1} H_{k3}$$

wher H is the Moore-Penros inverse of H

6 1 SQJ⌐Rᴇ COᴍPLᴇTION AND SOLUTION OF LIN AR EQUATIO

s the deconvolution problem is essent ally solving a s t of linear equations we first consider application of square compl tions for solving a system of linear equat ons given by

$$A\underset{\sim}{x} = b \qquad (6\ 5)$$

where A is an nxm matrix with n > m Let i nk of A(r(A)) be m Let $A_s = [A \quad A_{s1}]$ be a nonsingular square completion of A Such a choice of A_{s1} is possibl b cause A is a full rank ma rix Let $\underset{\sim}{x}$ be an ext nded vector

$$x_e = \begin{bmatrix} x \\ 0 \end{bmatrix} \qquad (6\ 6)$$

wh r o is an n-m length ze o v ctor In t rms of A_s^{-1} we can ob ain a solution to (6 5) given by

$$\tilde{x}_e = A_s^{-1}\ \underline{b} \qquad (6\ 7)$$

Note that $\tilde{\underline{x}}_e \neq \underline{x}_e$ in g neral We will return this point later Now let

$$\tilde{x}_e = \begin{bmatrix} \underline{x}_1 \\ x_2 \end{bmatrix} \qquad (6\ 8)$$

where x_1 is a solution o (6 5) with respect to he comple- t on A_{s1} Depending on th c oic of A_{s1} difr ient solutions are obtained natural quest on th d mands answer at this

stage is whether all such solutions are equivalent in some sense or not. Because if the solution given by (6 6) i independent of A_{s1} it may be economical in terms of computations to choose A_{s1} such that inversion of $_s$ involves least computational complexity. One simple result in this connection is the following

Theorem 5 1 If $\underline{b}$ belongs to the range space of A then $\tilde{\underline{x}}_e$ given by (6 7) is independent of A_{s1}

Proof Let $\underline{a}_o$ $\underline{a}_1$ $\underline{a}_{m-1}$ be the columns of A. Since $\underline{b}$ is in the range space of A $b = c_o\underline{a}_o + c_1\underline{a}_1 + \quad c_{n-1}\underline{a}_{m-1}$ for some c_i s. Therefore solution to (6) is given by

$$\underline{x} = (c_o \; c_1 \qquad c_{n-1})^T = \underline{c}$$

Let A_s be any nonsingular square completion of A. Let

$$_s^{-1} = \begin{bmatrix} A_{s2} \\ \bar{A}_{s3} \end{bmatrix}$$

Then from (6 8) we see that

$$\underline{x} = A_{s2}\,\underline{b} = \quad _{s2} \quad \underline{c}$$

Since $_s^{-1}A_s = I_n$ we obtain $A_{s2}A = I_m$. Therefore $\underline{x}_1 = \underline{c}$ which is independent of A_{s2} **

Note that the condition on b demanded by Theorem 5 1 cannot be guaranteed in practic i b does not belong to the range space of A. We consider this case in Section 6 3. In the next section we give a brief review of the conventional methods available for dealing with this case

6 2 LEAST SQUARE MI lI U NORM OLUTIO ID I OORE P NROSE GENEPALIZED INVERSE

The metho of solution normally used when $\underline{b}$ is not in the ra ge sp ce of A is to constrain the solution space The m thod of l ast quares is one way of constraining the solution p ce le choose only those x s which minimize

$$||\ x - \underline{b}||^2 = (\ x \quad b)^T\ (A\underline{x} - b) \tag{6 9}$$

when A is of rank m it can be shown that the right hand side of (6 9) is minimized by

$$x_L = (A^T A)^{-1}\ A^T b$$

when does not hav full rank ($r(A) < m$) the method used is to further constrain the solution space by considering $\underline{x}$ s hich minimiz (6 9) and have minimum length It can be shown tha such a solution is unique and is given by [85] [87]

$$\underline{x}_{mLS} = A^+\ \underline{b}$$

where A^+ is called the Moore-Penrose (MP) invers of A[85] A^+ s tisfies the following properti s

$$\begin{array}{lll} 1 & AA^+ & = A \\ 2 & A^+AA^+ & = A^+ \\ 3 & (A\ ^+) & = AA^+ \\ 4 & (A^+A)^* & = A^+A \end{array} \tag{6 10}$$

A matrix A^{LS} which satisfies the conaitions $AA^{LS}A = A$ and $(AA^{LS}) = AA^{LS}$ is called a least squ re inve se of [8]
When A is of full rank it can be shown that $A^{LS} = A^{+}$

3 L AST SQUARES SOLUTION VI SQUAR COMPLETIO S

Another approach that can be taken when b is not in the range space of is to e tend the solution space It is easy to s e that squ i completion is a w y of extending the solution space The solution obtained by forming $A_s^{-1}b$ depends on A_{s1} which controls the solution space In such situations it is in t uctive to know what kind or completions lead to lea t squares olution Alt natively we are int re ted in A_{s1} s which make A_{s2} in $A_s^{-1} = \begin{bmatrix} A_{s2} \\ A_{s3} \end{bmatrix}$ a lea t squares inver or Moore-Penrose inverse Th following lemma characterizes MP inverse of A via square completions

Lemma 6 1 Let $A_s = [A \quad A_{s1}]$ and $A_s^{-1} = \begin{bmatrix} A_{s2} \\ A_{s3} \end{bmatrix}$ Then A_{s2} is the MP inverse of A iff $(A_{s1}A_{s3})^{*} = A_{s1}A_{s3}$

Proof We know that $A_s^{-1}A_s = A_sA_s^{-1} = I_n$ Therefor in terms of the partitions of A_s and A_s^{-1} th following matrix equations must be satisfied

$$A_{s2}A = I_n \tag{6 11a}$$

$$A_{s2}A_{s1} = 0_{mxn-m} \tag{6 11b}$$

$$A_{s3}A = 0_{n-mxm} \tag{6 11c}$$

$$A_{s3}A_{s1} = I_{n-m} \tag{6 11d}$$

$$AA_{s2} + A_{s1}A_{s3} = I_n \tag{6 11e}$$

Jsing (6 11a) it is easy to show t A_{s2} satisfies the first three properties of he P-inverse of A given by (6 10) The ourth prop rty in (6 10) requires that

$$(\quad _{s2}) = \quad _{s2}$$

From (6 11e) we can se that this condition will b satisfied iff $(\quad _{s1} \quad _{s3}) = \quad _{s1} \quad _{3}$ *

Using L mma 6 1 we can prove the following theorem

Th orem 6 2 Given a nonsingular compl ion of A $A_s = [\quad A_{s1}]$ and it inverse $_s^{-1} = \begin{bmatrix} A_{s2} \\ A_{s3} \end{bmatrix}$ $_{s2}$ i th P invers of iff $A^T A_{s1} = O_{mxn-m}$

Proof Let A_{s2} be P inv rse of A Then $_{s2} = (A^TA)^{-1}A^T$ ubstituting for A_{s2} in (6 11b) we see that $A^TA_{s1} = O_{mxn-m}$ This proves the first part of the theorem

Now assume that $A^TA_{s1} = O_{mxn-m}$ Let (A) denote the r nge pace of A Then $R(A) \perp R(A_{s1})$ Here $\perp$ indicates that $R(A_{s1})$ is orthogonal to R(A) From (6 11b) we have $R(A^T_{s2}) \perp R(A_{s1})$ Since $R(A_{s1}) \perp R(A)$ it follows that the rows of A_{s2} pans R(A) But columns of A also sp n R(A) So $A_{s2} = DA^T$ for some nonsingular matrix D Subst tuting for A_2 in (6 11a) we get $D = (A^TA)^{-1}$ Tha implies $A_{s2} = (A^TA)^{-1} \quad ^T$ **

From now onward we shall r fer to square completions $_s$ having th property $A^- A_{s1} = O_{mxn-m}$ as orthogonal completions

In th next th orem v consider the impli ation o Theorem 6 1 in the ca e of k-Circulant completions

eorem 6 3 Giv n a nonsinqular k-Circulant co pletion of H $H_k = [H \quad H_{k1}]$ and its inverse $H_k^{-1} = \begin{bmatrix} H_{k2} \\ H_{k3} \end{bmatrix}$ H_{k2} is the MP-inv rse of H iff H_k i an oithogonal k-Circulant atrix

Proof From the definition o k-Circulant atrix (Appendix B) v c ll that h_1 th irst column of H_1 is related to oth r columns or H_k through η_k which is a k Circulant matrix with first ro (0 0 k) n otherwords

$$h_i = \eta_k^{i-1} \underline{h}_1 \qquad i = 2 \ 3 \qquad n \qquad (6\ 12)$$

wh re $\underline{h}_i$ is th ith colum of H_k

Using Theor m 6 2 we know that H_{k2} i the MP inverse of H if $H^T H_{k1} = 0_{nxn-m}$

For this condition to hold ve must hav

$$\underline{h}_j^T \underline{h}_i = 0 \qquad j = 1\ 2 \qquad m \qquad i = m+1 \qquad n \qquad (6\ 13)$$

In view of (6 12) (6 13) implies

$$h_1^T (\eta_k^{j-1})^T \eta_k^{i-1} \underline{h}_1 = 0 \qquad j = 1\ 2 \qquad m$$
$$i = m+1 \qquad n$$

But $\eta_k^T\eta_k = k^2 I_n$ (see Appendix B)

Therefore we obtain $\underline{h}_i^T \eta_i^{-1} J h_1 = 0$

or $\eta_i^T \eta_k^1 h_1 = 0 \quad i = 1\ 2 \quad n-1$

But $\eta_k^1 \underline{h}_1$ is the i+1th column of H_k

Therefore H_k in this case is an orthogonal k-Circulant *

From the foregoing discussion it is clear that the square completion of rectangular matrices provide a general framework for the study of deconvolution problems Both the least squares approach and the method based on k-Circulant completion are particular cases with appropriate choice of the completions In the light of Theorem 6 3 in the next section we determine those channels which admit LS solution via k-Circulant completions

6 4 CHANNELS AMENABLE TO LS SOLUTION VIA k CIRCULANT COMPLETIONS

In this section we are interested in determining those channel convolution matrices H such that H_k is an orthogonal k-Circulant Two cases are to be considered

Case 1 $m > g$

In the case of block data transmission $m > g$ This is necessary because otherwise reduction in the effective data rate over the channel will be large The form of H_k in this case is shown below

$$H_k = \begin{bmatrix} h_o & 0 & & 0 & kh_g & kh_{g-1} & & kh_1 \\ h_1 & h_o & & & 0 & kh_g & & kh_2 \\ & h_1 & & & 0 & 0 & & \\ & & & h_o & & & & \\ h_g & & & h_1 & h_o & & & kh_g \\ 0 & h_g & & & h_1 & h_o & & 0 \\ & 0 & & & & & & \\ & & & & & & & \\ 0 & 0 & & h_g & h_{g-1} & h_{g-2} & h_1 & h_o \end{bmatrix}$$

It is easy to see that for H_k to be an orthogonal k-Circulant the following equations are to be satisfied

$$kh_oh_g = 0$$

$$k(h_oh_{g-1}+h_1h_g) = 0$$

$$k(h_oh_1+h_1h_2+ \qquad h_{g-1}h_g) = 0$$

Since $h_o \neq 0$ we obtain

$$h_g = 0 \quad h_{g-1} = 0 \qquad h_1 = 0$$

This means that the only channel that admits LS solution via k-Circulant completion is the distortionless channel having $h_o \neq 0$ and all other ISI components zero

/ now consid r the case n > m+g This case aris s in most of the deconvolution applications For instance when FFT is to be used for inverting H It is desirable to hav n a power of 2 in such situations Therefore wh r n is not a power of 2 the channel vector is pa ded with zeros Another motivation for consid ring this case is that ere may exist some n > m+g such th t LS solution is obtained via k-Circulant completion

In this case H_k ill have a more general form as shown b low

$$H_{ke} = \begin{bmatrix} H & H_{k4} \\ - & - \\ O & H_{k3} \end{bmatrix}$$

where O is a null matrix of order n m-gxm Using the arguments given earlier we can easily show that H_{ke} is an orthogonal k Circular t only when $h_o \neq 0$ and $h_i = 0$ $i = 1\ 2$ g

Case 2 m < g

This situation arises in convolution si uations where the number of unknowns is less than the number of impulse response coefficients

In this case H_k takes the following form when g = 4 and m = 2

$$H_k = \begin{bmatrix} h_o & 0 & kh_4 & kh_3 & kh_2 & kh_1 \\ h_1 & h_o & 0 & kh & kh_3 & kh_2 \\ h_2 & h_1 & h_o & 0 & kh_4 & kh_3 \\ h_3 & h_2 & h_o & h_o & 0 & kh_4 \\ h_4 & h_3 & h_2 & h_1 & h_o & 0 \\ 0 & h_4 & h_3 & h_2 & h_1 & h_o \end{bmatrix}$$

For H_1 to be an orthogonal k-Circulant must have

$$kh_o h + h_2 h_o + h_3 h_1 + h_4 h_2 = 0$$

$$kh_o h_3 + kh_1 h_4 + h_3 h_o + h_4 h_1 = 0$$

$$kh_o h_1 + kh_1 h_2 + kh_2 h_o + h_3 h_4 = 0$$

Any solution to the above set of equations serves as the desired channel. Thus for m < g we see that it may be possible to obtain LS solution via k-Circulant completion. Explicit form of h_i's is difficult to obtain in this case. In the next section we give an alternative characterization of orthogonal k-Circulant which helps in obtaining some insight into the nature of the possible solutions.

6.5 CHARACTERIZATION OF ORTHOGONAL k-CIRCULANTS

From Appendix B we have

$$H_k = F_k^{-1} \operatorname{diag}(\lambda_o \; \lambda_1 \; \ldots \; \lambda_{n-1}) F_k = F_k^{-1} \Lambda F_k \qquad (6.14)$$

and $\Gamma_k = FD$ where $D = \mathrm{diag}(1\ p\ p^2 \quad p^{n-1})$

Using these in

$$H_k H_k = I_n \tag{6 15}$$

we obtain

$$F^{*} \Lambda F_k^{-1*} \Lambda F_k = I_r$$

or $\quad D F^{*} \Lambda^{*} {}^{-1*} D^{-1} D^{-1} F^{-1} \Lambda F D = I_n$

where $\Lambda = \mathrm{diag}(\lambda_o\ \lambda_1 \quad \lambda_{n-1})$

Noting that $F = \Gamma^{-1}$ we obtain

$$\Lambda F (DD^{*})^{-1} F^{-1} \Lambda F = D^{*-1} D^{-1} \tag{6 16}$$

or $\quad \Lambda^{*} F(DD^{*})^{-1} F^{-1} \Lambda = F(DD^{*})^{-1} F^{*}$

Let $\quad X = F(DD^{*})^{-1} F^{-1}$ (6 17)

Then $\quad \Lambda^{*} X \Lambda = /$ (6 18)

The problem of determining orthogonal k Circulant thus reduces to the determination of Λ such that (6 18) is satisfied for the given k If none of the elements of / are zero due to the diagonal nature of Λ it is easy to see that λ_1 s must satisfy the condition

$$\begin{bmatrix} \lambda_o \\ \lambda_1 \\ \\ \lambda_{n-1} \end{bmatrix} [\lambda_0\ \lambda_1 \quad \lambda_{n-1}] = E \text{ (matrix of all ones)} \tag{6 19}$$

To see thıs lⁿt n = 2 Thⁿn will be of the form

$$X = \begin{bmatrix} x_{11} & _{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$\text{then} \quad \Lambda X \Lambda = \begin{bmatrix} \lambda_1^2 \, x_{11} & \lambda_1 \lambda_2 \, x_{12} \\ \lambda_2 \; \lambda_1 x_{21} & \lambda_2^2 \, x_{22} \end{bmatrix}$$

In order to satısfy (6 18) we must have

$$\lambda_1^2 = 1 \qquad \lambda_1 \lambda_2 = 1 \quad \text{and} \quad \lambda_2^2 = 1$$

The procedure for obtaınıng an ortho jon l k-Cırculant ıs as follows lor the gıven v lue of k (6 18) mu t be solved to obtaın λ_1 s which ın turn produce H_k vıa (6 14) As an llustratıon we con ıder he case when l = 1

When k = 1 $(DD^T)^{-1}$ becomes an ıdentıty matrıx Hence

$$X = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & & 0 \end{bmatrix}$$

U ıng (6 18) we s e tha λ_1 s for an or hogonal cırculant satısfy

$$\lambda_1^2 = 1 \qquad \lambda_2 \lambda_n = 1 \qquad \lambda_3 \lambda_{n-1} = 1 \qquad\qquad \lambda_n \lambda_2 = 1$$

For a real matrix H_1 w have

$$\lambda_i = \lambda^*_{n-i+2} \qquad i = 2\ 3 \qquad n \quad \text{with } \lambda_{n+1} = \lambda_1$$

Therefore

$$|\lambda_i|^2 = 1 \quad i = 1\ 2 \qquad M \quad \text{where } M = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Different choices for $\lambda_1\ \lambda_2 \quad \lambda_M$ decide different orthogonal real circulants Thus we have infinit numb r of orthogonal Circulants The λ_i s are given by

$$\lambda_i = \cos\theta_i + j\ \sin\theta_i \qquad 0 \leq \theta_i < 2\pi$$

For instance when $n = 4$ we can have

$$\lambda_1 = 1 \quad \lambda_3 = 1 \quad \lambda_2 = \cos\theta + j\sin\theta$$

Then the first row of the desired k-Circulant is given by

$$h_1^T = (1+\cos\theta \quad \sin\theta \quad 1-\cos\theta \quad \sin\theta)$$

6 6 LS SOLUTION USING k-CIRCULANT COMPLETION

From the analysis in the previous sections ve conclude that for most cases of practical interest LS solution to the deconvolution problem cannot be obtained via k-Circulant completions as given by (6 1) However although he solution given by (6 1) is not the LS solution it may be very lose to the LS solution In this section we obtain an alternate expression for the LS solution using k-Circulant completions

Consider the received vector r given by

$$r = Hx + w \tag{6 20}$$

We first obtain the LS solution using heuristic arguments If w were to be a zero vector and H_k^{-1} exists for some k we can obtain exact solution to (6 20) as

$$x = x_1 = H_{k2}\, r \quad \text{and} \quad \underline{x}_2 = H_{k3}\, r = 0 \tag{6 21}$$

where

$$H_k = [H \quad H_{k1}] \qquad H_k^{-1} = \begin{bmatrix} H_{k2} \\ H_{k3} \end{bmatrix} \qquad x_e = \begin{bmatrix} x \\ o \end{bmatrix} = \begin{bmatrix} x_1 \\ \underline{x}_2 \end{bmatrix}$$

On the otherhand when $\underline{w}$ is not a zero vector we have in general $x \neq x_1$ and $\underline{x}_2 \neq o$ Now consider x_2 given by $\underline{x}_2 = H_{k3}\, \underline{r} = H_{k3}(Hx + w)$ But $H_{k3}H = 0_{n-mxm}$ Therefore

$$H_{k3}\, \underline{r} = H_{k3} \tag{6 22}$$

The minimum norm least square solution (6 22) is given by

$$\hat{w} = H_{k3}(H_{k3}\, H_{k3}^{*})^{-1}\, H_{k3}\, \underline{r} \tag{6 23}$$

Thus we have estimated the noise vector $\underline{w}$ in the best possible manner under the given conditions We can use this estimate to reduce the effect of noise in (6 20) Substituting for $\underline{r}$ using (6 20) in (6 21) for x_1 we obtain

$$\underline{x}_1 - H_{k2}\, r = \underline{x} + H_{k2}\, \underline{w}$$

Using (6 23) for $\underline{w}$ we get

$$H_{k2}\, r = x + H_{k2}\, H_{k3}(H_{k3}\, H_{k3}^{*})^{-1}\, H_{k3}\, \underline{r}$$

or $$\underline{x} = H_{k2}[I - H_{k3}^{*}(H_{k3}\, H_{k3})^{-1}\, H_{k3}]\, r$$

We now claim that no better result can be achieved under the given conditions A formal proof of this claim is given below

Theorem 6 4 $$H^{+} = H_{k2}[I-H_{k3}(H_{k3}\, H_{k3}^{*})^{-1}\, H_{k3}] \qquad (6\ 24)$$

Proof We show that (6 24) satisfies all the four properties of the MP-inverse given by (6 10) Using (6 11) where instead of A_s we have H_k and H_l^{-1} replace A_s^{-1} with $H_{k1} = A_1$ $H_{k2} = A_{s2}$ and $H_{k3} = A_{s3}$ it is easy to show that representation of H^{+} given by (6 24) satisfies the following conditions

$$HH^{+}H = H$$

$$H^{+}HH^{+} = H^{+}$$

and $$(H^{+}H)^{*} = H^{+}H$$

We have to finally show that $(HH^{+})^{*} = HH^{+}$ For this we make use of (6 11e) which gives after appropriate substitutions

$$HH_{k2} + H_{k1}H_{k3} = I_n$$

Therefore $$HH^{+} = HH_{k2}[I-H_{k3}^{*}(H_{k3}H_{k3}^{*})^{-1}\, H_{k3}]$$

$$= [I-H_{k1}H_{k3}][I-H_{k3}(H_{k3}H_{k3}^{*})^{-1}\, H_{k3}]$$

$$= [I-H_{k3} (H_{k3}H_{k3})^{-1} \quad _{k3}] - {}^{-}{}_{1\,1}H_{k3} + {}^{4}{}_{1}H_{k3}$$

$$= [I-H_{k3}^{*} (H_{k3}\, H_{k3})^{-1} H_{k3}]$$

which is a symmetric matrix

Therefore $(HH^{+}) = HH^{T}$

But such a matrix is unique [35]

$**$

In the foregoing discussion we used heuristic arguments for obtaining (6 24) Now we will show that (6 24) can b derived in a more formal manner For this purpose we write (6 20) using k-Circulant completion of H and extended vector x_e as

$$\underline{r} = H_k \, x_e + w \qquad (6\ 25)$$

Note that $\underline{x}_e$ is an extended version of x obtained by padding $\underline{x}$ with zeros However while seeking solution to (6 25) via H_k^{-1} we ignored this constraint on x_e Therefore the solution naturally differs from x_{LS} except in some special cases This means that in order to obtain LS solution to (6 25) we must minimize

$$(\underline{r} - H_k\, x_e)^{*} (\underline{r} - H_k\, \underline{x}_e) \qquad (6\ 26)$$

subject to the constraint

$$C\, x_e = 0 \qquad (6\ 27)$$

where $C = [O_{n-mxn-g} \quad I_g]$

This minimization problem can be solved using Lagrange multiplier method. Let $\underline{\alpha} = (\alpha_1\ \alpha_2\ \ \ \alpha_g)^T$ be the multipliers. Then we form

$$F(\underline{x},\ \alpha) = (\underline{r}-H_k\ \underline{x}_e)^*\ (\underline{r}-H_k\ \underline{x}_e) + 2\alpha^T\ C\ \underline{x}$$

Differentiating $F(\underline{x},\ \alpha)$ with respect to x and α we obtain conditions for minimum as

$$H_k^*\ H_k\ \underline{x}_e - H_k^*\ \underline{r} + C^T\ \alpha = 0$$

and $$C\ x_e = 0$$

As H_k is invertible we obtain

$$\underline{x}_e = H_k^{-1}\ \underline{r} - H_k^{-1}\ H_k^{*-1}\ C^T\ \alpha$$

and $$\underline{\alpha} = (C\ H_k^{-1}\ H_k^{*-1}\ C^T)^{-1}\ C\ H_k^{-1}\ \underline{r}$$

Existence of $(CH_k^{-1}H_k^{*-1}C^T)^{-1}$ follows by noting that

$$CH_k^{-1} = H_{k3} \quad \text{where} \quad H_k^{-1} = \begin{bmatrix} H_{k2} \\ H_{k3} \end{bmatrix}$$

and H_{k3} is a full rank matrix. Thus

$$\underline{x}_e = H_k^{-1}\ \underline{r} - H_k^{-1}\ H_k^{*-1}\ C^T\ (H_{k3}\ H_{k3}^*)^{-1}\ H_{k3}\ r$$

Therefore the solution to the original problem is given by

$$\underline{x} = H_{k2}\ \underline{r} - H_{k2}\ H_{k3}^*\ (H_{k3}\ H_{k3}^*)^{-1}\ H_{k3}\ r$$

U ing Th or m 6 4 we see that x is the LS solution to th deconvolution problem

Using (6 24) we obtain an xpression for the di er nc $\Delta H = H_{k2} - H^{+}$ given by

$$\Delta H = H_{k2} \, H^{*}_{k3} \, (H_{k3} \, H_{k3})^{-1} \; H \qquad (6\ 28)$$

Note that ΔH is ind pendent of H^{+} Therefore it c n be computed by knowing H_{k}^{1}

If the l ments o H are small the solutions obtained using H_{k2} and H^{+} will not differ s gnificantly In th case of orthogonal k-Circul nt completions of H it is easy to see that $\Delta H = O_{mxn}$ as $H_{k2} \, H_{k3} = O_{mxn-m}$

In summary we hav studi d LS d c nvolution problem in the framework of squar completion of rectangular matrices We have shown that in most of the practic l situations LS solution cannot be obtained via dir ct inv rsion of k Circulant completion of the chann l convolution matrix

Another method of obtaining LS solution is by using it rative schemes In the next chapter we study steepest descent algorithm in th discrete Fourier transform domain which provides LS solution to the deconvolu ion problem

CHAPTER 7

L ST SQUAR S DECONVOLUTION - ITERATIV APPROACH

In the previou chapter we considered th pos ibility of applying computat onally attractive k-Circulant completions of th ch nnel matri H for obtaining L solution to the de onvolu ion p oblem which involves inversion of H^TH Our analysis showed that in most of the practical ases the solution given by (6 1) using l-Circulant completions do not yield the LS solution to the deconvolution problem n alternate expression for L solution was also obta ned in S ction 6 6 Though th derivation of this expression mak s use of k-Circulant completions the final expr ssion is not computa ionally attractive compared to the direct inversion of H^TH

In this chapter we study ano her metho of obtaining LS solution to the deconvolution problem It is based on the application of wellknown steepest descent algorithm (SDA) [89] which is an iterative technique in contrast to the noniterative or direct approach considered in Chapter 5

Gersho [24] appears to have used ixed step SDA for the first tim for the purpos of adaptive chan el equ lization in which the tap oefficients of the equalizer are adjusted continuously so that the mean square error between the actual and

stimated data symbols is minimized He has shown that in the or sence of noise the algorithm yields a suboptim l setting of the tap coeffici nts

Walzman and Schwartz [21] [2_] have studied fixed step sup in the DFT dom in for the pu pose of a tom ti channel equalization Usin two examples they have observed that th DFT domain SDA conv rge faster than the sample domain SDA as exact knowledg of th eigenvalu are avail ble in th case of DFT domain SDA Their work has be n extended by Picchi and Prati [64] who hav includ d self orthogonal zing feature into the DFT do a n i ration scheme Using simulation studies th y have shown tha the per ormance o self-orthogonalizing DFT domain qualization scheme is sup r or to that of the sample domain algorithm cu to Godard [68]

The observations mad by Walzman and Schwartz [21] and Picchi and Prati [64] ar more carefully ex mined in this chapter One re son for this is based on our und rstanding that DFT b ing an orthonormal transformation th it ra ive sch mes in both the domains must have same convergence properties But there may be computational advantages in going rom sample domain to th DFT o main due to th availability of fast algorithms for computing DFT

We devot a major portion of this chapter towards a b tter understanding of the role of DFT and k-Circulant completions in

rcducing processing tin ano oroviuing rast convcrg nc Our naly is clearly shows how DFT helps in redjcing conputa ions Jsing th fr me jorl o l~Circulart comolctions w show that dvant ge in t r or f ster converg nc r t can bc chi v a in the DFT domain b cau it p ovides i1 th c sc o some xamples good stirate of the con/erg 1c arar ter it i ho m th t th cstimate obtaincd using k-Circulart c molction can also be used in the sample aon in to obtain s me conv r-g nc ra e

Th chaotcr is orgaiz d in the followirj mann r A ori f r vi of steepest d cent algorithn is given in section 7 1 lhc n nne n which DFI helps in r ducinc computat ons is dis-cuss d in S ction 7 2 In section 7 3 we aeri/e SDA in th DFT domain from thre c f i nt vi wpoints oat rccovcr/ scheme basea on DFT dome n SDA is iv n in scction 7 3 1 Convcrg nce oropertics of SDA r discuss d in Sc tion 7 4 method of increasing the onvera nce rate of the SD is dis cus d n Section 7 5 We show ho / Circulant or k-Circulan comolr ions c n be usca to r duc numbei or iter tions This analysi orovides xplanation for the f st convcrgence beha-vior o he self-orthogcnalizinc DFi donain scheme studied by Picchi ana Prati [64] In cti 7 6 wc show ho stimat s of step length in the cas f iix a tep SDA can be obtained using eigenv lues of th k-Circulan completions This explains the obscrvation mad by / l-nan nd Schuarcz [21] reqarding

fa t conv rgence of DFT domain scheme compa ed to sample domain sc re

1 REVIEW OF STEEPEST DESCENT LGORITHM

In this s ction w discuss briefly c rtain g neral f atures of he wellknown Cau hy s steep st d scent lgorithm [89] Consid r th qu drati obj ctiv function giv n by (2 8)

$$F(x) = x^T H^T H\underset{\sim}{x} - 2x^T H^T r - r^T r \qquad (7\ 1)$$

The SD constructs a sequenc of v ctors starting with x_0 chos n a bitrarily given by

$$\underset{\sim}{x}_{i+1} = \underset{\sim}{x}_i + \alpha_i H^T(\underset{\sim}{r} - Hx_i) \qquad i > 0 \qquad (7\ 2)$$

wh r α_i is det rmined such that $F(x_{i+1})$ is minimized Substituting (7 2) in (7 1) and diff rentiating w r to α_i it can be shown that α_i is given by

$$\alpha_i = \frac{e_i^T e_i}{e_i^T H^T H\underset{\sim}{e}_i} \qquad \underset{\sim}{e}_i = H^T(r - H\underset{\sim}{x}_i) \qquad (7\ 3)$$

Computation of α_i at ev ry iteration is the most time con sum ng part of SDA Ther fore in most of th practical application α_i i fix d at a value α chosen such th t th optimum speed of c nverg nce is achieved It can b shown [89] that the iteration scheme (7 2) conv rges to th LS solution

$$x_{LS} = (H^T H)^{-1} H^T \underset{\sim}{r}$$

I may be int rest ng to note th t the fixed step SDA ($\alpha = \alpha$ for all i) is essentially a generalized version of the Jacobi iteration [74] for solving the linear e uation

$$H^T H x = H^T r$$

The iteration scheme in this cas ta es he fo r

$$x_{i+1} = (I_m - \alpha H^T H)\ x_i + \alpha H^T r \tag{7 4}$$

This iteration scheme can be shown to converge f $0 < \alpha < 2/\lambda_{max}$ Th op i ium value o α is given by

$$\alpha = \frac{2}{\lambda_{max} + \lambda_{min}} \tag{7 5}$$

where λ_{max} and λ_{min} are the maximum and minimum eigenvalues of $H^T H$ respectively [90] (see page 24)

7 2 APPLICATION OF DFT TO STEEPEST DESCE T ALGORITHM

W now consider the possibility of using DFT to reduce computational load associated with SDA Consider e_i given by

$$e_i = H^T(r - Hx_i)$$

Let H_1 be the Circul nt completion of H and x_{ie} be the extended vector formed by padding x_i with n-m zero Then w can write

$$e_i = H_1^T\ (r \quad H_1\ x_{ie}) \tag{7 6}$$

where $e_i = \begin{bmatrix} e_i \\ - \\ e_{i1} \end{bmatrix}$ and $e_{i1} = H_{11}^T(r - H_1 x_{ie})$ $H_1 = [H \quad H_{11}]$

From (7 6) it is clear that if we compute the ri ht hand s de of (7 6) $\underline{e}_1$ is obtained as he first n ntries of $\underline{e}_1$ Now let $\underline{E}_1$ denot the DFT of e_1 In erm of the DFT of r $\underline{x}_e$ and h the first column of H_1 we can write (7 6) in th DFT domain as

$$\underline{E}_1 = FH_1^T F (Fr - FH_1 F Fy_1$$

$$= F_1 \quad (\quad H_1 \quad {}_{1e}) \tag{7 7}$$

Th r fore by knowing H_1 P and $_1$ the r h hand side of (7 7) is easily computed and e_1 is then obt ined as

$$e_1 = PF^* \underline{E} \tag{7 8}$$

wher $P = [I_m \mid O_{mxr-m}]$

Now if $\underline{E}_{1e}$ denotes th DFT of $\underline{e}_{1e}$ where e_{1e} is an extended vector obtained by padding $_1$ with (n-m) z ros α_1 given by (7 3) can be written as

$$\alpha_1 = \underline{e}_1^T e_1 / \underline{E}_{1e}^T (F_1^* H_1 \underline{E}_{1e}) \tag{7 9}$$

Thus by knowing the DFT of e_{1e} computation of α_1 becomes relatively simple task as we already have the knowledge of $\underline{H}_1$

It should be noted that the value of α_1 remains same both in the sample domain and th DFT domain Accordingly the

convergence rate remains unaltered by the application of DFT We will com back to this point later

Further we wish to note that as k-DFT computation using FFT algorithm requires more computation compared to DFT k-Circulant completions other than k = 1 are not as useful in reducing computations per iteration compared to Circulant completion

7 3 DERIVATION OF THE DFT DOMAIN STEEPEST DESCENT ALGORITHM

In this section we derive SDA in the DFT domain from three different viewpoints The objective is to clearly bring out the role of DFT in SDA

Approach 1 Using the extended vector $\underline{x}_e$ and Circulant completion H_1 of H we can write (7 2) alternatively as

$$\underline{x}_{(i+1)e} = \underline{x}_{ie} + \alpha_i \, P \, H_1^T (\underline{r} - H_1 \underline{x}_{ie})$$

$$\text{where} \qquad P = \begin{bmatrix} P \\ O_{n-m \times n} \end{bmatrix} \qquad (7\ 10)$$

Premultiplying the above equation by the DFT matrix we obtain

$$\underline{X}_{(i+1)e} = \underline{X}_{ie} + \alpha_i FP \, H_1^T (r - H_1 \underline{x}_{ie})$$

where $X_{ie} = Fx_{ie}$ Since $F^{*}F = I_n$ we can modify the above expression as

$$\underline{X}_{(i+1)e} = \underline{X}_{ie}+\alpha_i FP\ F^{*}\ \ H_i^T F\ (F\underline{r}\ FH_i F\ Fx_{ie}) \tag{7 11}$$

Now FH_iF^* and $FH_i^TF^*$ are diagonal m trices with DFT of th first column of H_i and its complex conju ate a diagonal elements respectively Th refore we can writ (7 11) s

$$X_{(i+1)e} = X_{ie} + \alpha_i FP\ F\ H_i^{*}\ \ (\underline{P}-\underline{H}_i\ \underline{X}_{ie}) \tag{7 12}$$

Moreover we can write (7 12) also as

$$\underline{X}_{(i+1)e} = \underline{\ }_{ie} + \alpha_i\ \underline{E}_i \tag{7 13}$$

where

$$\underline{\ }_i\ = FP\ F^{*}\ \ _i \tag{7 14}$$

$$\underline{E}_i\ = \underline{H}_i\ (\underline{P}\ -\ \underline{H}_i\ \underline{X}_{ie})$$

α_i can be expr ssed as

$$\alpha_i\ = \underline{E}_{ie}^T\ \underline{E}_{ie}/\ \underline{E}_{ie}^T(H_i^{*}\ \underline{H}_i\ E_{ie}) \tag{7 15}$$

The pair (7 13) and (7 15) repre ents DFT domain i ration schem s corresponding to (7 2) and (7 4) respectively

Approach 2 In Approach 1 w derived DFT domain SDA starting from sample domain SDA In the pre ent approach we formulate the deconvolution p oblem as

$$\min_{\underline{z} \in R^n} \quad (r - H_1 z)^T (\underline{r} - Hz) \tag{7 16}$$

subject to $Cz = 0$

$$\text{where} \quad C = [O_{n-m \times m} \quad I_{n-m}] \tag{7 17}$$

The minimization problem given by (7 16) involves symmetric Circulant positive semidefinite matrix in general It may be noted that eventhough $H^T H$ is positive definite $H_1^T H_1$ need not always be so A simple illustrative example is given below

Consider

$$H = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{clearly} \quad H^T H = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ is positive definite}$$

whereas the Circulant completion of H

$$H_1 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{gives} \quad H_1^T H_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

which has 0 as an eigenvalue Consequently $H_1^T H_1$ is positive semidefinite

It may be further noted that whereas minimization of $F(\underline{x})$ given by (7 1) is an unconstrained problem (7 16) defines a constrained minimization problem Such problems can be solved

using Rosen approach of projecting the gradient at every iteration on to the null space of C [89], [91]. Let G be a matrix such that $CG = 0_{n-m,m}$. From (7.17) it is easy to see that G can have the following form

$$G = \begin{bmatrix} I_m \\ -- \\ 0_{n-m,m} \end{bmatrix}$$

Following the approach given in [89] (see pages 260-266) we can write down the iteration scheme for (7.16) as

$$z_{i+1} = \underline{z}_i + \alpha_i G G^T H_1^T (r - H_1 \underline{z}_i) \tag{7.18}$$

where $\alpha_i = \min_{\alpha} F(\underline{z}_{i+1})$. But $GG^T = P$. We can now use DFT and write (7.18) as done in the case of Approach 1 in the DFT domain. Thus we obtain

$$Z_{i+1} = \underline{Z}_i + \alpha_i FP\, F^*(\underline{H}_1^t\, r - \underline{H}_1^*\, \underline{H}_1\, Z_i) \tag{7.19}$$

Since $\underline{Z}_i$ plays the same role as $\underline{X}_i$ in (7.13) we see that (7.13) and (7.18) are identical.

Approach 3: We now consider the approach taken by Walzman and Schwartz [21] and Picchi and Prati [64] for deriving DFT domain SDA. Using F and the Circulant completion of H we can rewrite the minimization problem given by (7.16) as

$$\min_{y \in R^n} (F\underline{r} - FH_1\underline{y})^{*T} FF^*(Fr - FH_1\underline{y})$$

writing $FH_1\underline{y} = FH_1F^*F\underline{y}$ and noting that $FF^* = I_n$ we obtain

$$\min_{Y \in C^n} \; (\underline{R}-H_1\underline{Y})^{*T}(R-H_1Y) \qquad (7\ 20)$$

where by our earlier convention $R = Fr$, $H_1 = F\underline{h}$ and $\underline{Y} = F\underline{y}$. Now in terms of $Y \varepsilon C^n$ we can formulate the minimization problem given by (7 16) equivalently as

$$\min_{\underline{Y} \varepsilon C^n} \; (P-H_1\underline{Y})^{*T}(\underline{R}-H_1Y) \qquad (7\ 21)$$

subject to 1) F^*Y is real vector

and 2) $CF^*Y = 0$

where C is same as in (7 16) Solution to the above problem can be obtained using Rosen s gradient projection method as described in [21] Alternatively since (7 21) is derived using (7 16) which has (7 19) as the corresponding SDA in the DFT domain it follows that (7 21) also will have (7 19) as its corresponding SDA in the DFT domain

At this stage we wish to note the following distinctive features of the three approaches Approach 1 deals with a unconstrained minimization problem In Approach 2 we deal with a constrained minimization problem with one constraint In the third approach we have to deal with two constraints The form of SDA in the DFT domain however remains unchanged Likewise

he convergence parameter α_1 also remains unchang d by he application of DFT Thus we can conclude ha DFT and th associated Circulant co pl tion help only in reducing the computational compl xity of th iterativ scheme These aspects donot b cone clear ir the SDA i d rived using Approach 3 alone because solution of the minimization probl n given by (7 21) do s not sh w explicitly its relationship with SDA in th ample domain

7 3 1 Data R covery Schem based on SDA in th DFT Domain

A DR scheme based on DFT domain iter tion sch m (7 13) is sch matically sho n in Figure 7 1 In this scheme the starting point s chosen as

$$\underline{X}_{oe} = \underline{R}|\underline{H}_1$$

Note that in the absence of noise and when the Cir ulant completion of H_1 is nonsingular $\underline{X}_{oe}$ repres nts he DFT of the actual data vector padded with n-m zeros Therefore in such situations the iteration scheme w ll converge in one iteration On the otherhand when noise is present $\underline{X}_{oe}$ provides a good starting point or he iteration s hem The p rformance of this scheme will o same as that of DR sch me based on LS solution

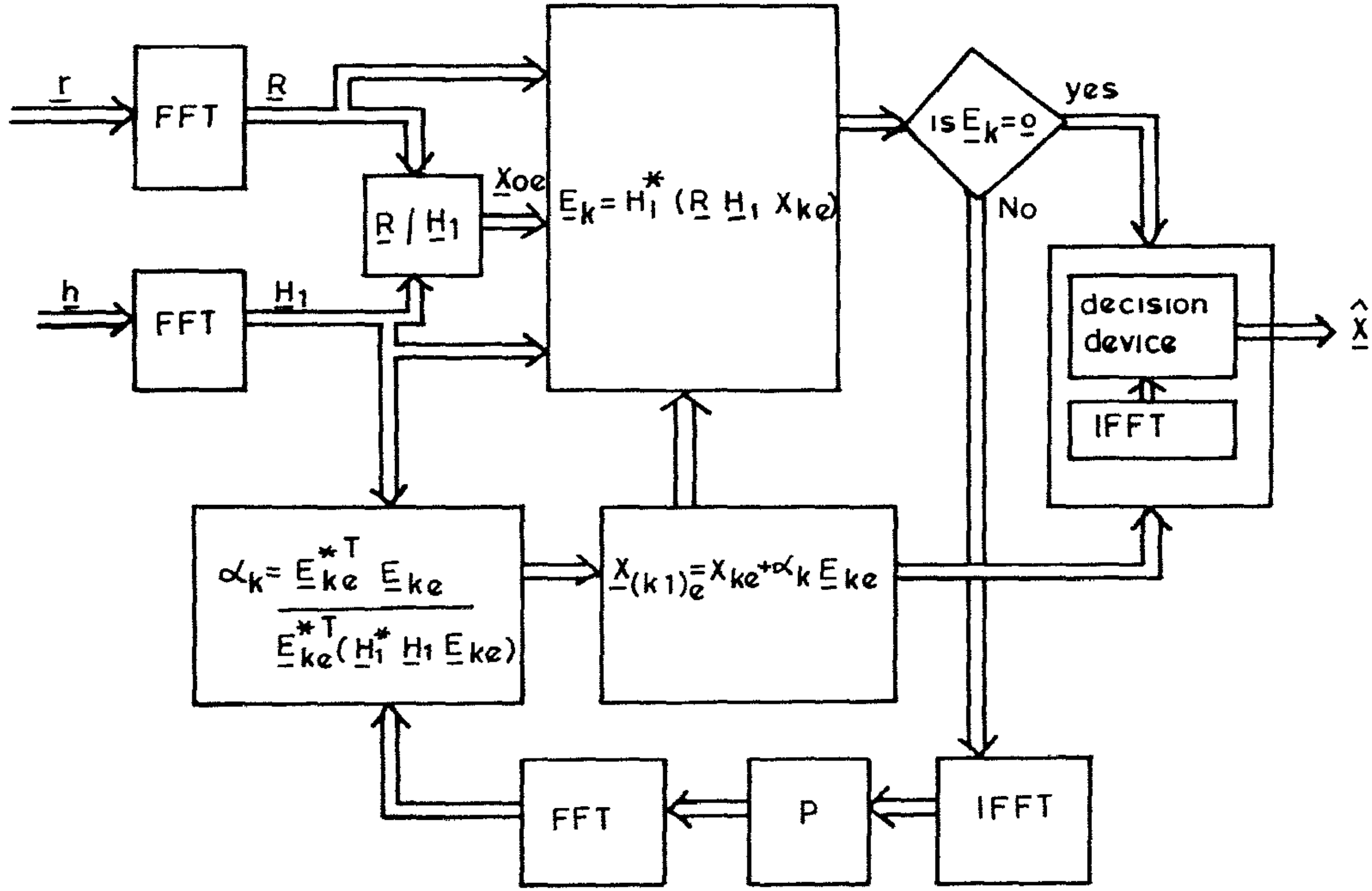


FIG 71 Data recovery scheme based on DFT domain steepest descent algorithm

7 4 RATE OF CONVERGENCE OF SDA

SDA generates an infinite sequence of $\underline{x}_i$ s whose limit is the solution to the optimization problem Due to severe constraints on processing time in a communication situation large number of iteration will not be a desirable feature Therefore it becomes important to know the speed of convergence of SDA In this connection we have the following theorem [89]

Theorem 7 1 Suppose that $\tilde{\underline{x}}$ is the solution to minimization of $F(\underline{x})$ given by (7 1) Let c_n be the condition number of H^TH That is $c_n = \lambda_{max}/\lambda_{min}$ Then (see [89] page 146)

$$F(\underline{x}_{i+1}) - F(\tilde{x}) \leq \left(\frac{C_n-1}{C_n+1}\right)^2 [F(x_i) - F(\tilde{x})]$$

$$||x_{i+1} - \tilde{\underline{x}}|| < C_n^{1/2} \left(\frac{C_n-1}{C_n+1}\right)^2 ||x_i - \tilde{\underline{x}}||$$

From Theorem 7 1 it is clear that SDA has linear rate of convergence The rate of convergence depends on the condition number of the channel autocorrelation matrix If $C_n = 1$ the iteration scheme converges in one iteration But as C_n increases the convergence rate decreases and hence large number of iterations will be needed before meaningful estimate of the solution vector is obtained Thus SDA is not an efficient method when the condition number of the channel is large One method of accelerating the convergence rate of SDA is by

altering the eigenvalues of the channel autocorrelation matrix so that condition number is near to unity In the next section we consider this aspect

7 5 FAST CONVERGING STEEPEST DESCENT ALGORITHM

Consider the iteration scheme given by (7 2) The scheme converges to the exact solution when

$$H^T(r - H\underline{x}_i) = \underline{O} \tag{7 22}$$

for some i This means that instead of α_i we can substitute αQ where Q is a nonsingular matrix and obtain another iteration scheme

$$\underline{x}_{i+1} = \underline{x}_i + \alpha Q H^T(r - H\underline{x}_i) \tag{7 23}$$

which will also give the same solution as (7 2) provided α Q and H satisfy the necessary convergent conditions We can show that the iteration scheme (7 23) converges if

$$||I_m - \alpha Q H^T H|| < 1$$

or $0 < \alpha < 2/\lambda_{max}$ where λ_{max} is the maximum eigenvalue of QH^TH

If $Q = (H^TH)^{-1}$ it is easy to see that the iteration scheme (7 23) converges to the solution in one iteration if we choose $\alpha = 2/(\lambda_{max} + \lambda_{min})$ where λ_{min} is the minimum eigenvalue of QH^TH (when $Q = (H^TH)^{-1}$, $\lambda_{min} = 1$)

It is clear from the above discussion that different choices of Q will lead to different iteration schemes. For $Q = I_m$ we obtain fixed step SDA. Since computation of the best choice for Q which is $(H^TH)^{-1}$ itself is uneconomical we must choose Q such that it is a good estimator of $(H^TH)^{-1}$ or the inverse of the channel autocorrelation matrix. Such choices are difficult to obtain in the sample domain.

Now let us look at the DFT domain representation of (7.23) given by

$$\underline{A}_{(i+1)e} = \underline{A}_{ie} + \alpha F Q_1 F^* E_{ie} \qquad (7.24)$$

where as before $\underline{Y}_{ie} = F\underline{x}_{ie}$ and $\underline{E}_{ie}$ is given by (7.14). Q_1 is now an nxn matrix. From (7.24) we observe that if FQ_1F^* is a diagonal matrix additional computation needed for multiplying FQ_1F^* and $\underline{E}_{ie}$ is just n multiplications. Therefore in order to make FQ_1F^* diagonal Q_1 must be a circulant matrix. Since the best choice for Q is $(H^TH)^{-1}$ it seems natural to choose for Q_1 $(H_1^TH_1)^{-1}$ whenever H_1 is nonsingular. This feature has been exploited by Picchi and Prati [64] in their fast converging self orthogonalizing DFT domain equalization scheme in the context of channel equalization.

So far in this section we discussed the application of Circulant completion for obtaining good estimates of the convergence parameter. However it may not be always possible

to obtain good estimate of the conv rgence parameter with Circulant completion In such cases k-Circulant completions can be us d For this purpose let $Q_k = (H_k^* H_k)^{-1}$ We assum that H_k i invertible When $|k| = 1$ it can be shown that Q_k is also a k Circulant Therefore making use of k-DFT (7 24) can b written using k Circulant completion of H a

$$\underline{X}_{(i+1)ke} = \underline{X}_{ike} + \alpha F_k Q_k F_k^{-1} \underline{E}_{ik}$$

where $\underline{X}_{ike} = F_k \underline{x}_{ik}$

and $\underline{E}_{ike} = F_k P \; F_k^{-1} \underline{E}_{ik} \quad \underline{E}_{ik} = \underline{H}_k^* \; (\underline{P}_k - \underline{H}_k \; \underline{X}_{ike})$

Since Q_k is a k-Circulant matrix $F_k Q_k F_k^{-1}$ will be a diagonal matrix It may b noted that use of k-Cir ulant completion r quires computation of k-DFT which can be computed u ing FFT algorithm as described in Chapter 5 Th efficiency of the iteration scheme depends on the value of k u ca a direct nalysis of th eigenvalue behaviour of Q_k and consequently its effect on α is difficult as it depends upon the channel given This aspect needs mor investigation

7 6 FIXED STEP STEEPEST DESCENT ALGORITHM

Th DFT domain version of fixed step SDA follows from our discu sion in Section 7 3 In th s case we can write (7 13) as

$$\underline{X}_{(i+1)e} = \underline{X}_{ie} + \alpha \underline{E}_{ie} \qquad (7\ 25)$$

The b st choice of α is given by (7 5) As comput t on of λ_{max} and λ_{min} is a difficult task stimates of th igenvalu s of the channel autocorrelation matrix can be used An alternative method would be to choose the eigenvalues of $H_1^T H_1$ or $H_k^T H_k$ if th y are good estimate or the eigenvalues of $H^T H$ One reason for such a choice is that they can b omputed easily using DFT Walzman and Schwartz [1] have used eigenvalues of $H_1^T H_1$ in the iterative scheme However th y do not seem to have noted that α obtained using $H_1^T H_1$ can also be used in the sample domain

Let $\lambda_{max\ k}$ and $\lambda_{min\ k}$ be maximum and minimum igenvalues of $H_k^T H_k$ and let

$$\alpha^{(k)} = \frac{2}{\lambda_{max\ k} + \lambda_{min\ k}} \qquad (7\ 26)$$

If $\alpha^{(k)}$ is close to α th degradation in the convergence rate o th iterative scheme will not be appreciable Towards this end we prove the following theorem

Theorem 7 2 Let λ_{max} and λ_{min} be the maximum and minimum eigenvalues of $H^T H$ Then $\lambda_{max\ s} \geq \lambda_{max}$

$$\lambda_{min\ s} \leq \lambda_{min} \qquad (7\ 27)$$

where $\lambda_{max\ s}$ and $\lambda_{min\ s}$ represent maximum and minimum eigenvalues of $H_s^T H_s$ where H_s is a nonsingular square completion of H

Proof We first prove that $||H^TH|| \leq ||H_s^TH_s||$ where $||\ ||$ is the spectral norm of a matrix Let $P = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}$ be an nxn matrix Using P we can write

$$||H^TH|| = ||PH_s^TH_sP||$$

for $||PH_s^TH_sP|| = \left|\left|\begin{bmatrix} H^TH & 0 \\ 0 & 0 \end{bmatrix}\right|\right| = ||H^TH||$

Therefore $||H^TH|| < ||P||^2\ ||H_s^TH_s||$

As P is a projection operator it has eigenvalues either 0 or 1

Therefore $||H^TH|| < ||H_s^TH_s||$

If $\lambda_{max\ s}$ and λ_{max} are the maximum eigenvalues of $H_s^TH_s$ and H^TH respectively we obtain

$$\lambda_{max} \leq \lambda_{max\ s}$$

That proves the first inequality in (7 27)

We now show that $||(H^TH)^{-1}|| \leq ||(H_s^Th_s)^{-1}||$

we have $H_s^TH_s = \begin{bmatrix} H^TH_c & H^TH_c \\ H^TH_c & H^TH_c \end{bmatrix}$

Note that $H_s = [H\ \ H_c]$ Consider the matrix G

$$G = \begin{bmatrix} I & 0 \\ -(H_c^TH)(H^TH)^{-1} & I \end{bmatrix}$$

Then $||G^{-1}|| = 1$ Therefore

$$||(H_s^T H_s)^{-1}|| = ||(H_s^T H_s)^{-1}|| \; ||G^{-1}||$$

Using the result $||AB|| \leq ||A|| \; ||B||$ we obtain

$$||(H_s^T H_s)^{-1}|| \geq ||(H_s^T H_s)^{-1} G^{-1}||$$

$$= ||(GH_s^T H_s)^{-1}||$$

Now $GH_s^T H_s = \begin{bmatrix} H^T H & H^T H_c \\ 0 & H_c^T H_c - H_c^T H (H^T H)^{-1} H^T H_c \end{bmatrix}$

Therefore $||(GH_s^T H_s)^{-1}|| \geq ||(H^T H)^{-1}||$

Therefore $||(H^T H)^{-1}|| \leq ||(H_s^T H_s)^{-1}||$

Therefore $\lambda_{min\ s} \leq \lambda_{min}$

where $\lambda_{min\ s}$ and λ_{min} are the minimum eigenvalues of $H_s^T H_s$ and $H^T H$ respectively **

In particular for k-Circulant completions we have

$$\lambda_{max\ k} \geq \lambda_{max}$$

and (7 28)

$$\lambda_{min\ k} \leq \lambda_{min}$$

In a given situation for fixing α it may be desirable to try out different k-Circulant completions and choose the one which provides maximum α But this involves additional computations The problem of maximizing α over all possible

k-Circulant completions for a given channel does not appear to admit mathematically tractable solutions Thi as ect however needs further investigation

In summary we have discussed the role o DFT and k-Circulart completions in reducing the comput tional complexity and increasing th convergence rate of SDA We have showed how to obtain better estimates of the convergence parameter α using eigenvalues of k-Circulart compl tions of channel convolution matrix H As there is a one-to one correspond nce between sample domain and DFT domain iteration chemes estimates of α obtained via k-Circulant completion can be used in the sample domain itself to prov de faster convergence

SDA has only linear converg nce property and the rate of convergence depends upon the condition number of the channel autocorrelation matrix In this connection elf orthogonalizing scheme discussed in Section 7 5 could be useful

Considering the drawbacks of SDA despite its several advantages it appears that when there is severe constraint on proces ing time a better method will have to be chosen Th method so chosen must converge fast and it must have a better rate of convergence We wish to note that conjugate gradient algorithm (CGA) [89] meets thes d mands Howev r it is computationally unattractive because each iteration takes more

processing tim compared to SDA Kobayashi [92] has considered CGA for adaptive channel equalization H cr CGA docs not app ars to have received much att ntion in the ar a f adaptive channel equalization du to its comput tional complexity

CHAPTEP 8

PERFO MANCE OF DATA RECOVE Y SCHE ES

In this chapter we give a comp rati/e ac oun of perfor- man e of some of the data r cover/ schemes tudied in previous hapters Performance is specified in term o p obabilit/ of error (Pe) Sinc the pr b bility of er r depends upon block length channel parameters at recovery s heme employed noise statistics and the rate of tran mission over h chann l except in very special cas s in general it is very difficult to obtain close form expressions fo Pe In thi connection it may be recalled that it is possible to obtain clos form expre- ssion for Pe for DR scheme on ider d in Chapt r 5 On the other hand th schemes consid red in Chapt rs and 4 are non- lin ar in nature and hence are di ficult to Under such situations simulation studies help ir evaluating the performance of the DP sch me employed Th oughout thi chapter we assume binary data transmission However th analysis can also be xt nded to multil vel data

This chapter is organized in the following mann r As a prelud to the study of th performan of various DR schemes in Section 8 1 we characterize good c annels for BDT which introduce ISI and provide good performance subject to a given

crit rion There are two re sons or kno ing u h channels Firstly they provide a benchmark for comparing h perfor m nc of other channels condly on coula u e or f lt r at the transmitter o as to make the ov rall r sponse of the channel nd th prefilter o resemble clo ely a good ch nnel Channel examoles used in th simul tion study are giv n in S ction 8 2 In S ction 8 3 w scuss performance of BDT sy t ns using p rformance curv s for a w a range of plock l ngths and typical chann l examples In S ction 8 4 we study the effect of increasing IsI among th numbers f a block which happ r s when the dat rate ov r the ch nn l is to be k pt the same as source d t rate

8 1 GOOD CHANNELS FOR BLOCK DATA TRA ISSIO

8 1 1 aximum Minimum Distanc Cnannels

L t S_x be the set of all possible m-l ngth d ta vectors We know from our discussion in Sec ion 2 4 that when r is received in the pres nce or hite Gaussian noise the best estimate of the transmitt d vector $\underline{x}$ is given by a vector $\underline{x}_1 \varepsilon S_x$ such that

$$||\underline{r}\ H\underline{x}\ || \leq ||r-H\underline{x}_J|| \quad \text{for all } \underline{x}_J \varepsilon S_x \tag{8 1}$$

L t S_y denote anoth r set defin a as

$$S_y = \{\underline{y}_J \mid y_J - Hx_J \quad \underline{x}_J \varepsilon S_x\} \tag{8 2}$$

wher y_j is the received vector corresponding to x_j when noise i absent Let d_{ij} denote the distance b tween two distinct vectors y_i and y_j Then

$$d_{ij} = ||y_i - y_j|| \quad i \neq j \tag{8 3}$$

Du to the presence of noise the deconvolution rule (8 1) need not result in x_i which i the actual d ta vector transmitted In ord r to minim e the o currenc of such situations it is d sirable to have good s paration b t n any pair of y_j s in terms of the distance metric given by (8 3) Let the minimum distance be d_{min}

$$d_{min} = \min_{y_i, y_j \in S_y} ||y_i - y_j|| \tag{8 4}$$

Without loss of gen rality we will consider channel respons s with unit length A g+1 length vector $\underline{f}$ is called maximum minimum distance (MD) channel sample response if

$$d_{min} = \min_{i \neq j} d_{ij} \quad 1 \leq i \; j \leq 2^m \tag{8 5}$$

subject to $\underline{f}^T\underline{f} = 1$ and $-1 < f_i < 1 \quad 0 \leq i < g$ is maximum

The econd onstraint in the above formulation is necessary to avoid trivial solutions for example $f_i = 1$ and $f_j = 0$ for $j \neq i \quad 0 \leq j \quad g$

d_{ij} in (8 3) can be written as

$$d_{ij}^2 = ||y_i - y_j||^2 = ||Hx_i \; Hx_j||^2 = (x_i \; x_j)^T H^T \; (x_i - x_j)$$

Since convolution op ration is commutativ w can int r-chang the rol s of x_i and h and write

$$d_{ij}^2 = f^T (X_i - X_j)^T \; (X_i - X_j) \; f \qquad (\quad)$$

wh re $f_i = h_i \qquad 0 \leq i \leq g$

and X_i is the convolution matrix with x_i in he first column
L t

$$Z_{ij} = X_i - X_j \qquad (8 \; 7)$$

It then follows that the elem nts of Z_{ij} are crnary variabl s (+2 0 -2) Moreover Z_{ij} is also a onvolution matrix
We can now write d_{ij}^2 alternatively as

$$d_{ij}^2 = \underline{f}^T \; Q_{ij} \; \underline{f} \qquad (8 \; 8)$$

wh re Q_{ij} is a symmetric Toeplitz matrix Q_{ij} is given by
$Q_{ij} = Z_{ij}^T \; Z_{ij}$
In terms of (8 8) the optimization problem (8 5) b com s

$$\max_{\underline{f} \in R^{g+1}} \; \min_{\substack{i \neq j \\ 1 \leq i \leq 2^m}} \; \underline{f}^T \; Q_{ij} f \qquad (8 \; 9)$$

subject to 1) $\underline{f}^T \underline{f} = 1$ (8 10)

and 2) $-1 < f_i < 1 \quad 0 \leq i \leq g$

Th following lemma provid s maximum value achi v ble if we solv (8 9) subject to (8 10)

Lemma 8 1   In the case of binary alphabet the maximum d_{min} achievable is at most 2

P oof   We make use of the inequ lity

$$\lambda_{min\ ij}\ \underline{f}^T\underline{f} \le \underline{f}^T Q_{ij}\underline{f} \le \underline{f}^T\underline{f}\ \lambda_{max\ ij} \qquad (8\ 11)$$

where $\lambda_{min\ ij}$ and $\lambda_{max\ ij}$ are the minimum and maximum eigen-values of Q_{ij}

Consider two data vectors which differ in only one posi-tion In this case Q_{ij} is a diagonal m trix with all diagonal entri s equal to 4 Therefore

$$\lambda_{min\ ij} = \lambda_{max\ ij} = 4 \qquad (8\ 12)$$

Using the condition $\underline{f}^T\underline{f} = 1$ in (8 11) we get

$$\lambda_{min\ ij} \le \underline{f}^T Q_{ij}\underline{f} \le \lambda_{max\ ij} \qquad (8\ 13)$$

From (8 12) and (8 13) it is clear that w cannot find an $\underline{f}$ for all $i \neq j$ such that

$$\underline{f}^T Q_{ij} f > 4 \quad \text{for all } i \neq j$$ **

Using Lemma 8 1 w prove th fol owing theorem

Theorem 8 1 Determination of $\underline{f}$ which gives $d_{min} - 2$ is equivalent to solving

$$f^T Q_{ij} f \geq 4 \quad \text{for all } i \neq j \text{ and } \lambda_{min\ ij} < 4 \qquad (\ \ 14)$$

$$\underline{f}^T f = 1$$

$$-1 < f_i < 1 \qquad\qquad 0 \leq i \leq g$$

Proof If $\underline{f}$ is chosen s ich that for all of those Q_{ij} with $\lambda_{min\ ij} < 4$ we have

$\underline{f}^T Q_{ij} \underline{f} \geq$ clearly for the res of the Q_{ij} s

$\underline{f}^T Q_{ij} \underline{f} > 4$ holds from (13)

**

According to Theorem 8 1 the proc oure for obtaining MMD channels is to first determine all those Q_{ij} s for which $\lambda_{min\ ij} < 4$ Then we have to solve the set of dratic inequalities given by (8 14) While good amount of literature is available on linear inequalit es [93] very little seems to have been achieved in understanding of systems of quadratic inequalities We will not solve this probler directly in this thesis Instead we will consider another characterization of good chann ls called maximum distance chann ls (MD-channels) le shall see later that MD-chann l haracterization is helpful in obtaining MMD channels

8 1 2 Maximum Distanc Channels

We no consider those ch nnel which maximize the sum of dis inct distances between all possible pairs of elements in S_y That i if the pairs y_i y_j and y_p y_q give the same distance ($d_{ij} = d_{pq}$) only one of the pairs will be considered in the summation In this case the problem may be formulated as

$$\max_{f \in R^{g+1}} \sum_{Q_{ij} \text{ distinct}} \underline{f}^T Q_{ij} \underline{f} \tag{8 15}$$

subject to (8 10)

put differently (8 15) can also be written as

$$\max_{f \in R^{g+1}} \underline{f}^T Q \underline{f} \tag{8 16}$$

subject to (8 10) where $Q = \sum_{Q_{ij} \text{ distinct}} Q_{ij}$ (8 17)

In the presence of the second condition in (8 10) the problem given by (8 16) is difficult to analy e We therefore solve the following simpler problem

$$\max_{\underline{f} \in R^{g+1}} \underline{f}^T Q f \tag{8 18}$$

subject to $f^T \underline{f} = 1$

A solution to problem (8 18) can be obtained by using (8 11) from which it follows that the eigenvector corresponding

to the maximum eigenvalues is the desired olution to (8 18) If this solution satisfies the second constr int in (8 10) then a solution to (8 16) is obtained provided th minimum distance achieved is 2 in the binary case However t proc dure outlined above is computationally unattractiv for large block sizes Moreover it has not been po sible to ont in conditions under which solution to (8 18) is also a solution to (8 16) We feel that a more rigorous study may lead to interesting result In the absence of such r sults we have followed the procedure outlined below for obtaining MD as well as MD channels

8 1 3 Procedure for Obtaining WMD and MD Channels

1 Given the block length m and number of ISI samples g form the Q matrix given by (8 17)

2 Determine the normaliz d eigenvectors and the correspond ing eigenvalues of Q

3 The eigenvector corresponding to the maximum eigenvalue is a solution to (8 18)

4 Compute the minimum distance for each eigenvector satisfying the second constraint in (8 10)

5 The eigenvector giving maximum minimum distance is a solution to (8 14) if the maximum minimum distance is 2 in the binary case

It must be noted that we may not get solution to (8.14) as given by the last step in all the cases. Towards this end we conducted some numerical experiments to obtain channel responses using the above procedure for several values of n and g. Some of these results are presented in Table 8.1. In the last column of the table we have given the minimum distances achieved. In all the cases we have obtained both MMD as well as MD channels. Another interesting feature is that some of the channel responses are reminiscent of the partial response channels [94]. For instance, in the case of two ISI terms the channel responses (.707 0 .707) and (-.707 0 -.707) are duobinary channel responses. The channels obtained for g = 4, 5, 6 also show similar features.

The foregoing observation is interesting because partial response signalling schemes were originally suggested in order to achieve better spectral shaping of the transmitted signal. As we do not have yet a formal definition of such schemes it is difficult to establish meaningful relation between spectral shaping and maximum distance separation. This aspect needs further investigation. At this point we wish to note that Folconer and Magoe [47] also have shown that optimum desired impulse of the channel and prefilter combination, of length two, under minimum mean square criterion must be the duobinary impulse response.

Table 8 1 Some examples of ıD and MMD Channels

m	g	Channel sampled response	d_{min}	Remarks
6	3	707 0 0 0 707	2	ıD and ıD
		- 707 0 0 0 707	2	D
6	4	707 0 0 707 0 0	2	ıı ıD nd D
		707 0 0 - 707 0 0	2	ı D
		0 0 - 707 0 0 - 707	2	ıD and D
		0 0 - 707 0 0 707	2	ı ıD
6	5	-0 8165 0 0 0 40825 0 0 0 40825	2	D
		0 57735 0 0 0 57735 0 0 0 57735	1 633	D
		0 0 707 0 0 707 0 0	2	ıı D
6	6	-0 8165 0 0 0 408 0 0 0 408 0 0	2	ıı D
		0 577 0 0 0 577 0 0 0 577 0 0	1 ꞇ◡3	D
7	7	-0 86603 0 0 0 2886 0 0 0 2886 0 0 0 2886	2	ı ıı ıD
		0 5 0 0 0 5 0 0 0 5 0 0 0 5	1 414	ıD

NMD channels with $d_{min} = 2$ have another interesting property when used in conjunction with L-d convolution or Viterbi algorithm For such channels there will be essentially no loss in SNR when ML-deconvolution m thod is adopted This conclusion is based on the analysis given by Forney [38] H has shown that in the cas of maximum lik lihood sequence estimation the symbol error probability is bound d by

$$K_L\, Q(d_{min}/N_o) < Pe \leq K_U\, Q(d_{min}/N_o) \tag{8 19}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\, dy \tag{8 20}$$

and K_L and K_U are constants independent of noise variance and the channel parameters K_L is typically within an order of magnitude of K_U. We know that for one shot transmission (no ISI) the probability of error is given by

$$Pe(\text{one shot}) = Q(2||\underline{f}||/N_o) \tag{8 21}$$

Thus if $d_{min} = 2||\underline{f}||$ the probability of error achieved in the presence of ISI using ML-deconvolution procedure will be approximately equal to Pe (one shot)

8 1 4 Computation of Minimum Distance of a Channel

A direct method of d termining d_{min} is to compute the distanc between all possible pairs of $\underline{y}$ s belonging to set S_y

given by (8 2) This is a difficult job even for moderate block lengths We now show that ML-PDA d v l d i Chapter 3 can be modified for computing d_{min}

Fr m our earlier discussion in this ch pt r it is clear that the problem of finding minimum distan e can be formulated as

$$\min_{\substack{x_i, x_j \in S_x \\ i \neq j}} (x_i - x_j)^T H^T H (x_i - x_j)$$

L t $\underline{z}_{ij} = \underline{x}_i - x_j$

Elements of $\underline{z}_{ij}$ are ternary variables taking values from the set $\{-2\ 0\ 2\}$ Let $\underline{z}$ denote a v tor whose el m nts are ternary variables In terms of $\underline{z}$ the above probl m can be alternately formulated as

$$\min_{\underline{z}} \underline{z}^T H^T H\ z \tag{8 22}$$

$$\text{subject to } z^T z > 0 \tag{8 23}$$

Note that the constraint $\underline{z}^T z > 0$ is neces ary to eliminate all zero solution for cas s corr sponding to $x_i = x_j$

The foregoing problem is a quadratic optimization problem in the presence of a constraint In order to find the non-zero minimum value we adopt the procedure of Chapter 3 with the following difference In the fir t step when z_1 is optimized for ev ry given value of z_2 z_3 z_{g+1} choose only non zero

z_1 s Re t of the recursion stage are c rried out wi hout any constraint This follows from the fact that s nce z_1 is constrained to take only nonzero values the constraint (8 23) is satisfied and hence rest of the z_1 s can be hosen arbitrarily We wish to note that in the context of CSDT Burkhart and Barbosa [95] have sugges ed a modified Viterbi algorithm for determining the minimum dis ance of a given hannel

8 2 CHANNELS USED IN THE STUDY

In order to compare the performance of diff rent data recovery schemes it is necessary to consider channels which show both merits as well as demerits of various DR schemes The channel examples considered are given in Table 8 2 In Table 8 2 the first two channels A1 and A2 are maximum distortion channels considered by Austin [13] The next two

Table 8 2 Chann ls used in the simulation s udy

Channel	sampled response	d_{min}
A1	$3^{-1/2}(1\ 1\ 1)$	1 633
A2	$5^{-1/2}(1\ 1\ 1\ 1\ 1)$	1 265
C1	$2^{-1/2}(-\ 235\ \ 667\ 1\ \ 667\ -\ 235)$	1 933
C2	$2^{-1/2}(\ 235\ \ 667\ 1\ \ 667\ \ 235)$	1 188
P1	(227 460 688 460 227)	1 131
P2	(407 815 407)	1 630

examples C1 and C2 taken from [4] Th 1 st two exa les P1 and P2 are taken from [7]

Maximum distortion channels as the name it elf indicates exhibit maximum realizable disto tion according to some distortion measure Austin [13] determined such channels by choosing the samples of the channel autocorrelation co ffici nts $a(-g)$ $a(0)$ $a(g)$ so as to maximize

$$\sum_{i=-g}^{g} |a(i)|$$

subject to 1 $a(0)$ is unity

and 2 $A(w) = \sum_{i=-g}^{g} a(i) \exp -jiw)$

$$1+2 \sum_{i=1}^{g} a(i) \cos(iw) > 0$$

when $w \varepsilon(0\ \pi)$

The second condition implies hat the channel is realizable Austin has shown that the sampled auto orrelation function having a triangular envelope corresponds to maximum distortion channels in the sense defined above

8 3 SIMULATION STUDIES ON DATA R COVERY SCHEMES

The following data recov ry schemes are considered

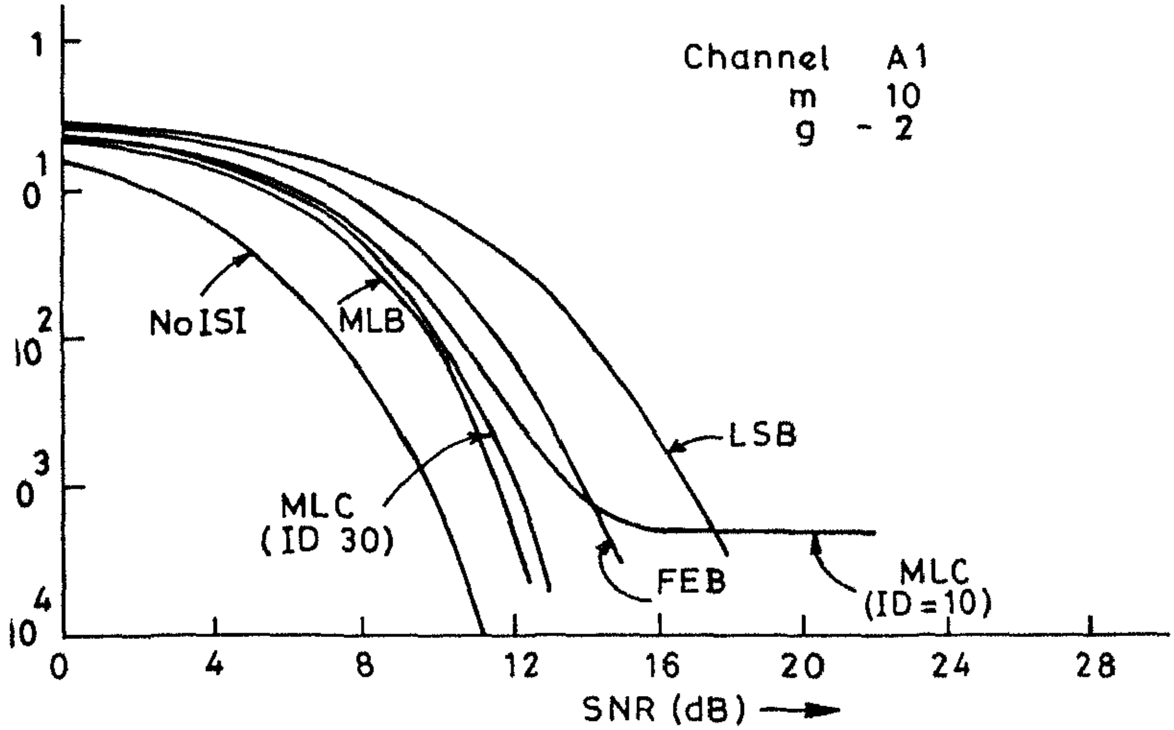
1 ML data recovery scheme for CSDT ? LC

2 ML data recovery scheme for BDT MLB

3	Decision feedba k scheme for CSDT	DFC
4	Modified Austin s approach for BDT	MAB
5	Fourier expansion method with decision reedback	FEB
6	LS data recov ry scheme for BDT	LSB
7	Circulant completion ba ec che e for BDT	CCB
8	Skew Circulant completion bas d ch me for BDT	SCB

Note that a three letter identirier is associ t d with each data recovery scheme these identifi rs have been used throughout in the text tables and figures We wish to note that LSB is based on Fourier expansion method without decision f edback In all the simulations 10000 data symbols taking binary (-1 1) values were used The curves plotted are obtained using single simulation

In Figures 8 1 - 8 9 w have shown the variation of th probability of error with SNR for the six channels given in Table 8 2 Performance curves in th case of MLC are obtained using ML-RDA given in Section 3 2 with the following modification D cision regarding a transmitted data is made after a fixed no of recursions say ID That is aft r the IDth stage we follow the path corresponding to the maximum value of the objective value achieved so for nd choose the data value orresponding to first stage After that we shift the ID stage results one stage back so that the second stage now becomes the



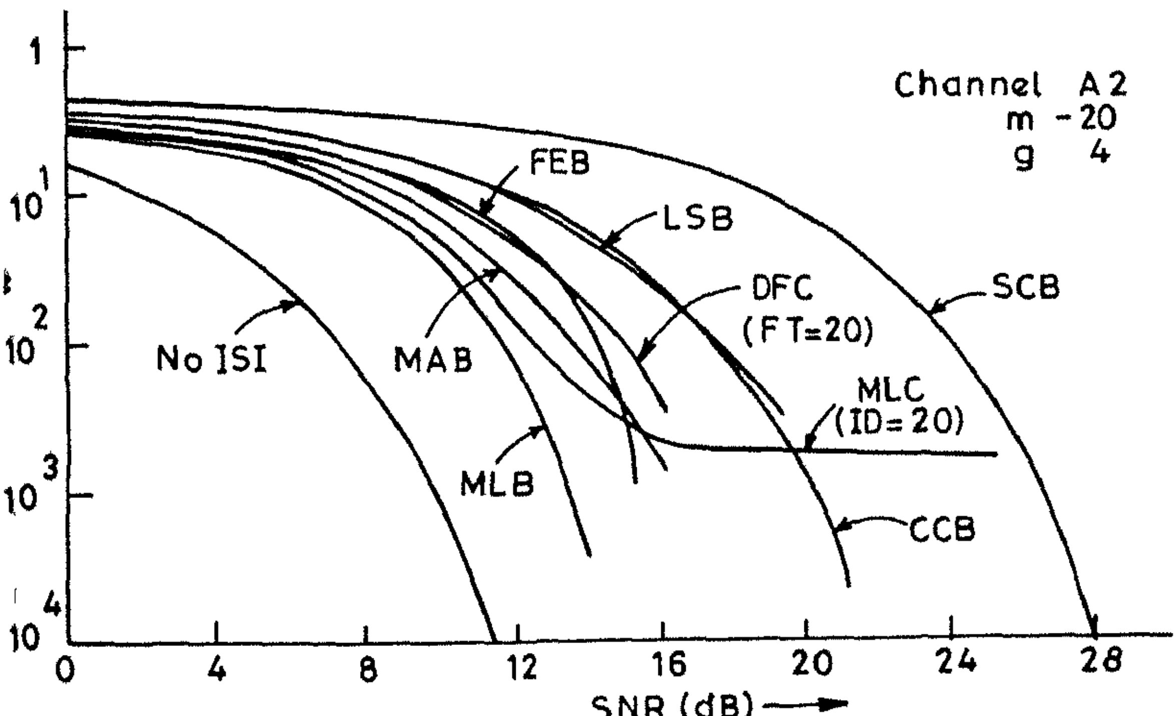
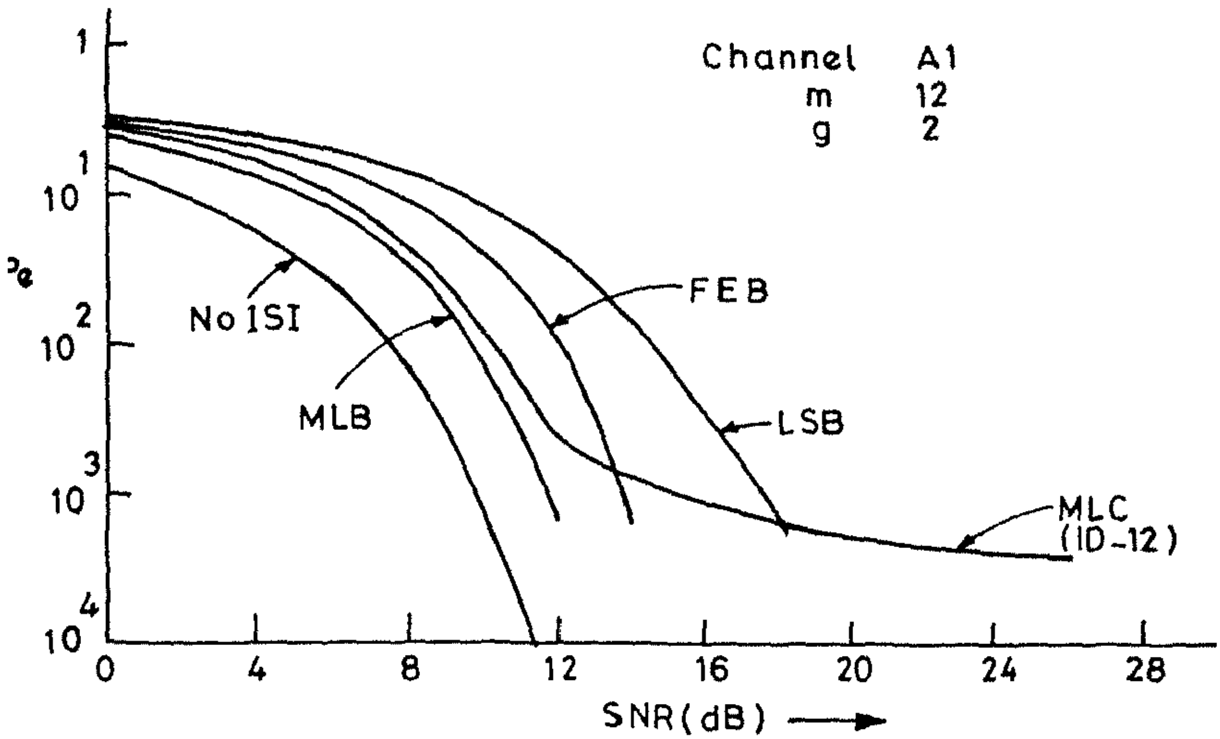


FIG 8 1 Comparison of performance of various DR scheme



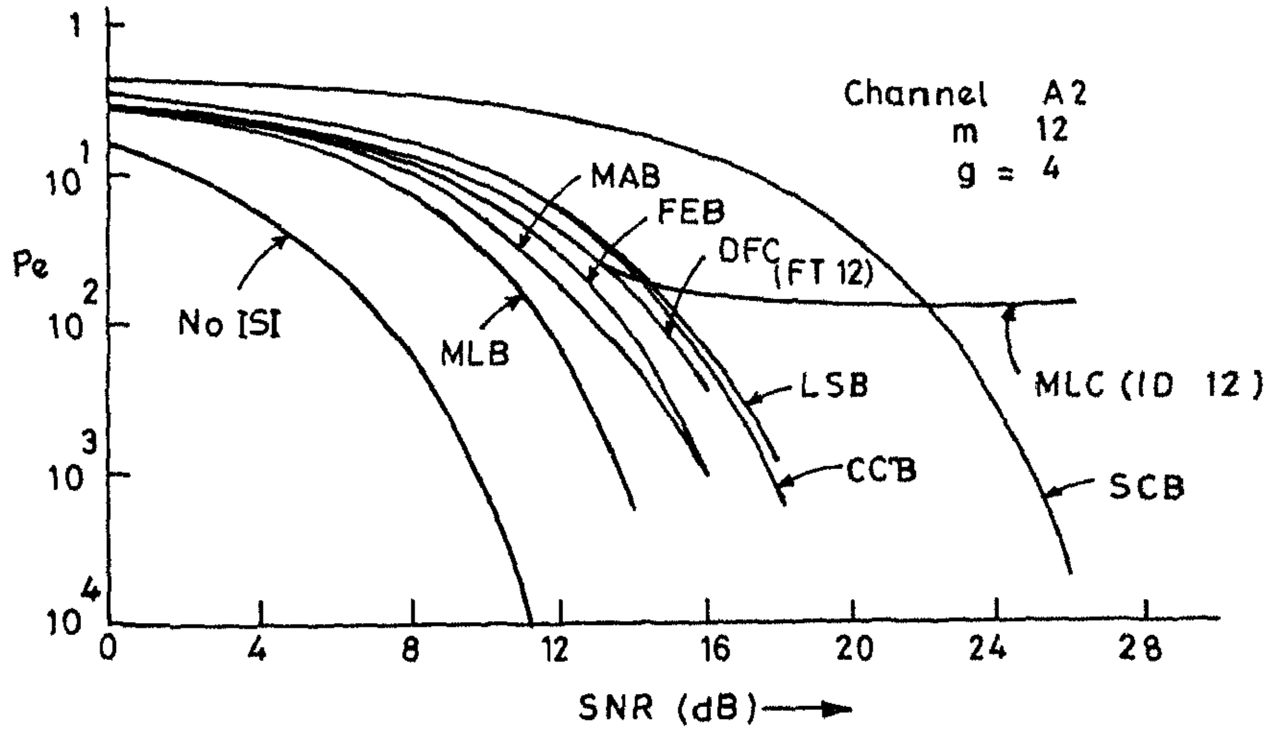


FIG 8 2 Comparison of performance of various DR schemes

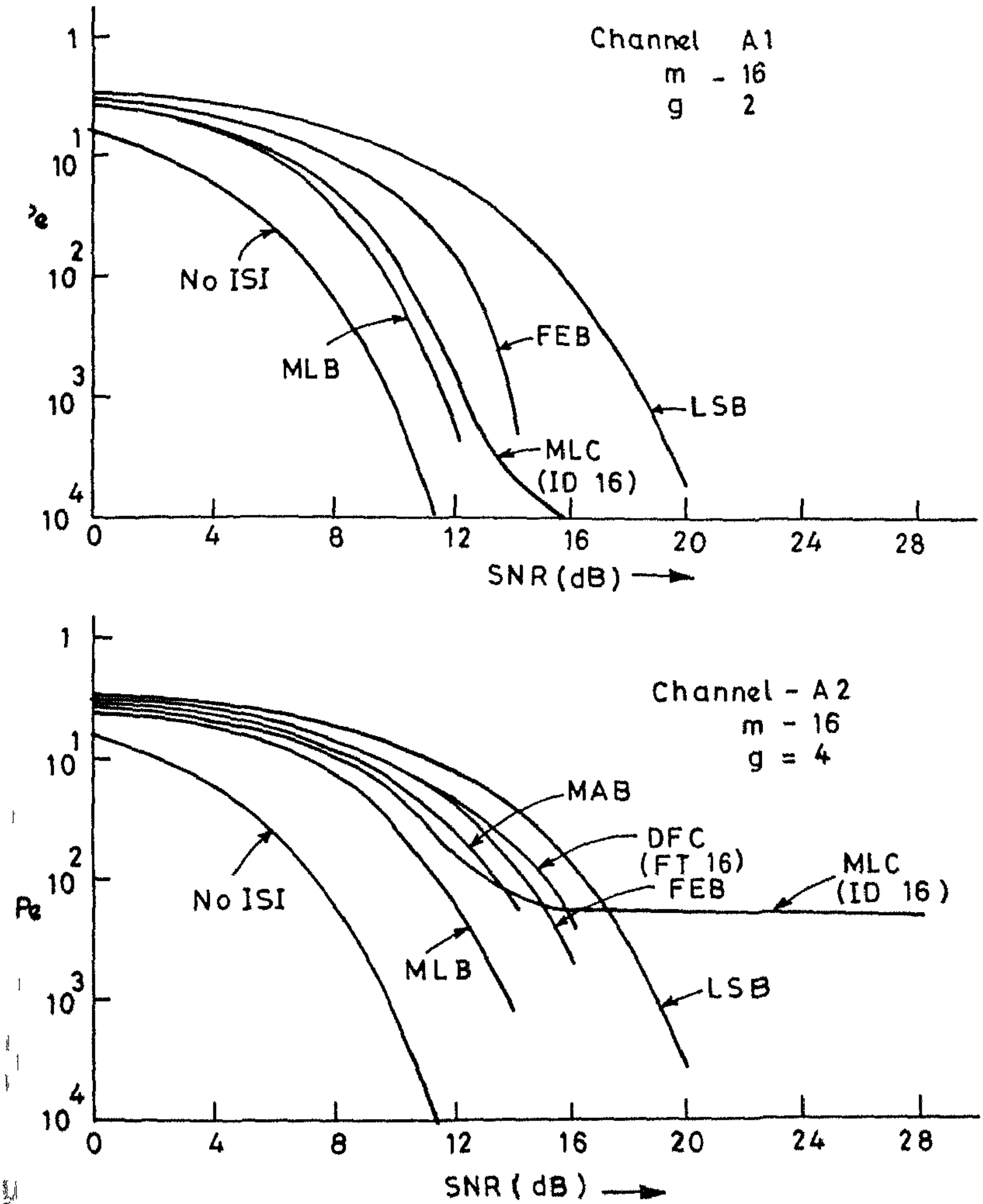


FIG 8 3 Comparison of performance of various DR schemes

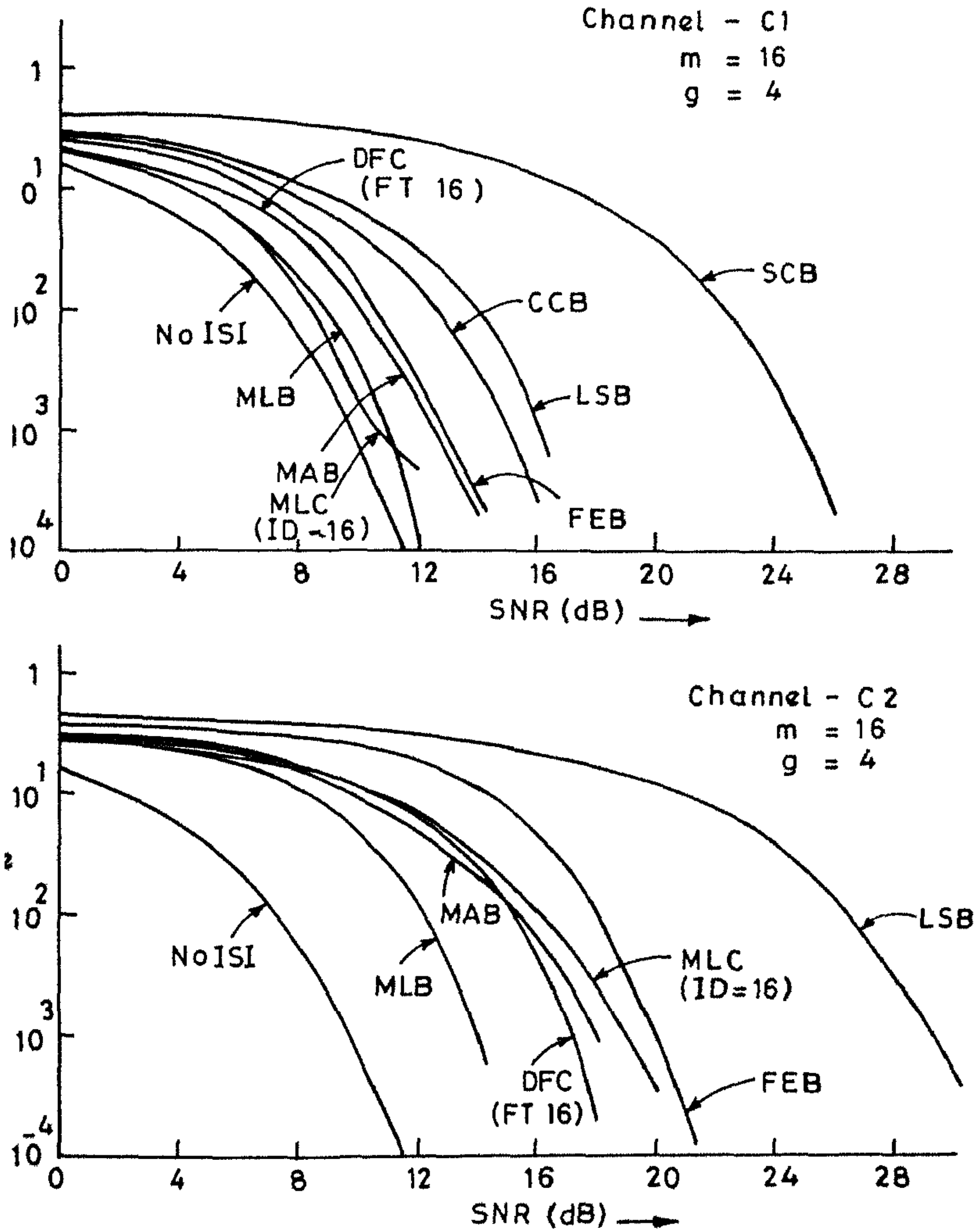
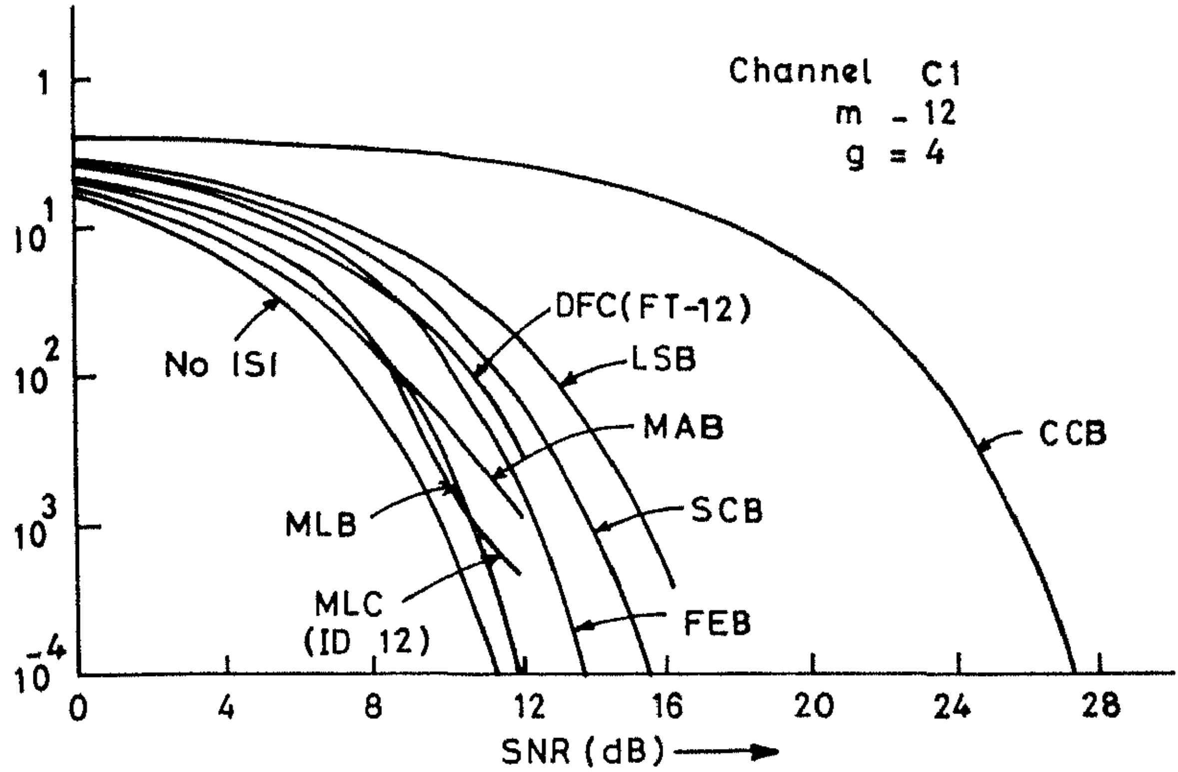


FIG 8 4 Comparison of performance of various DR schemes



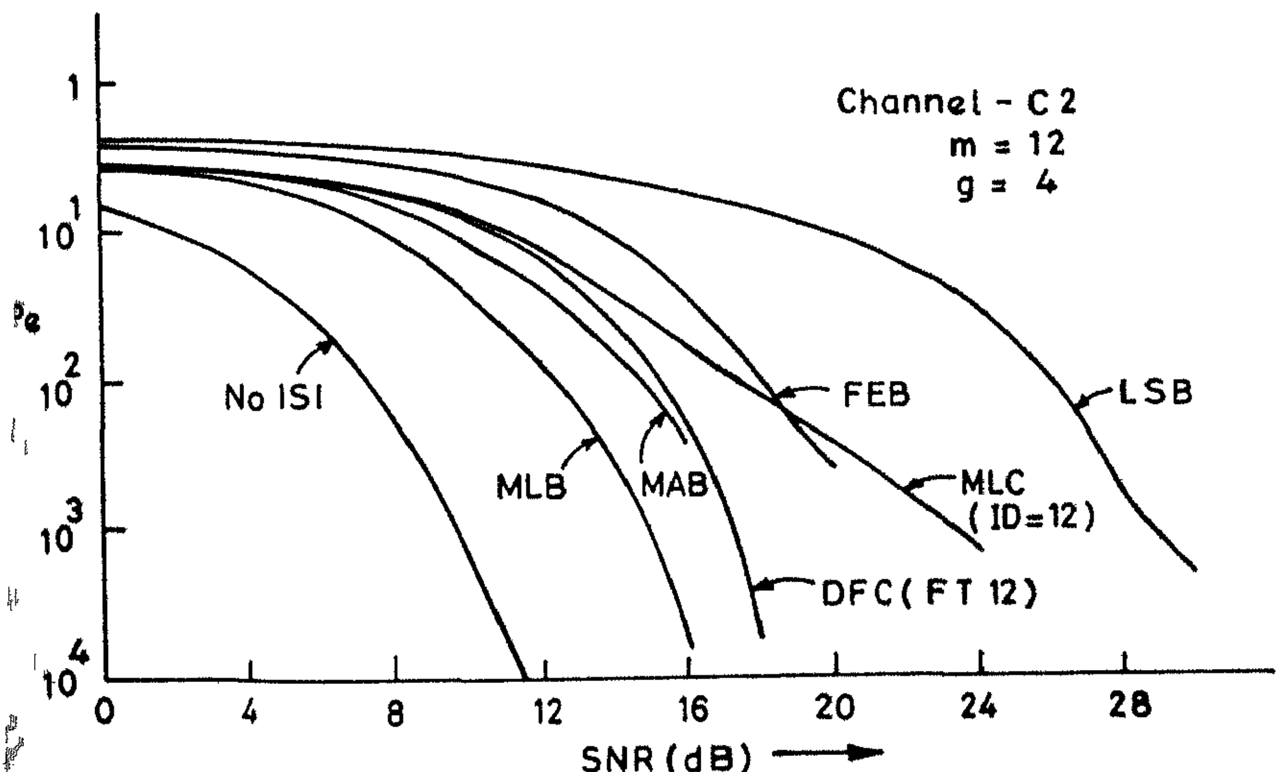


FIG 8 5 Comparison of performance of various DR schem

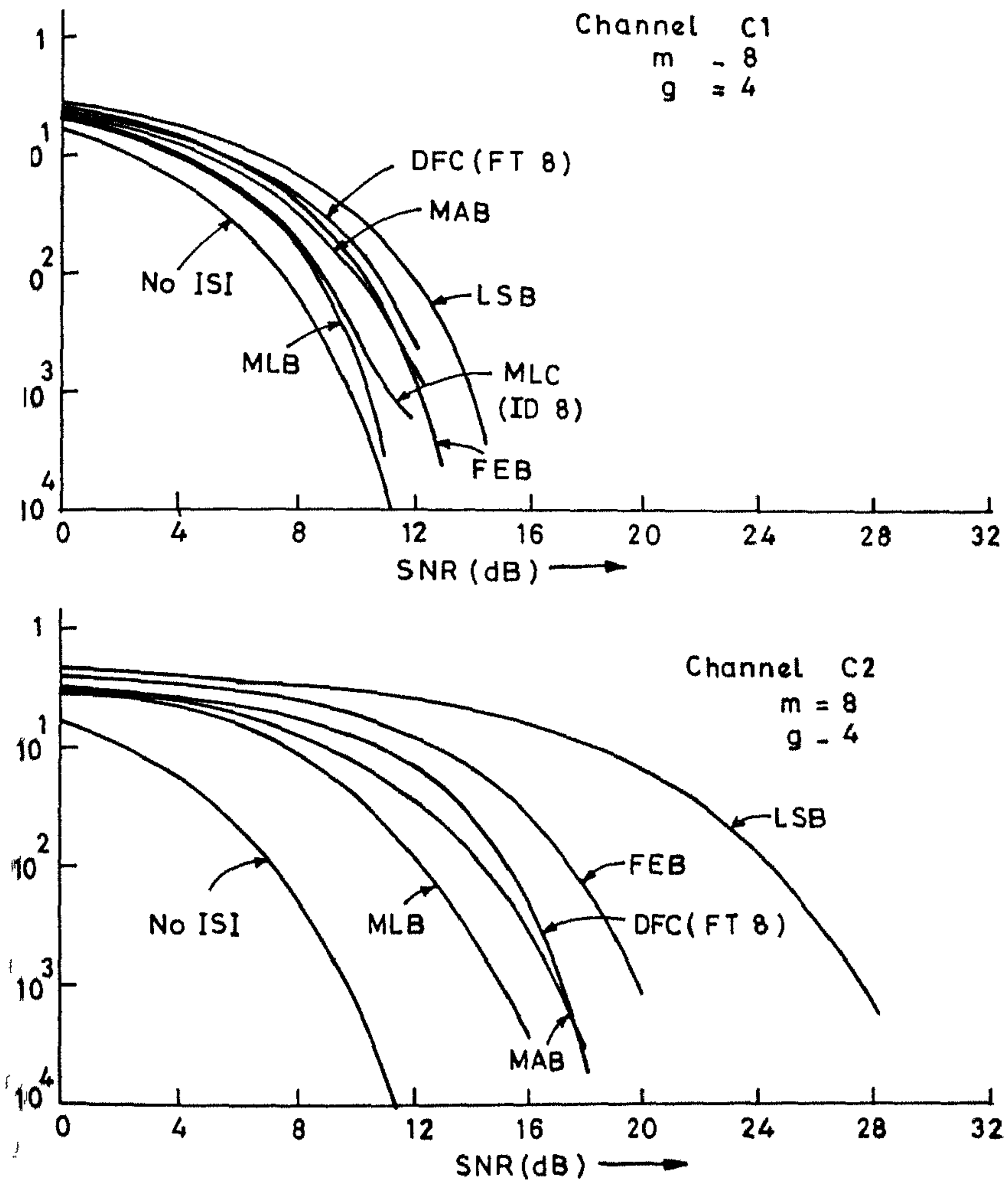
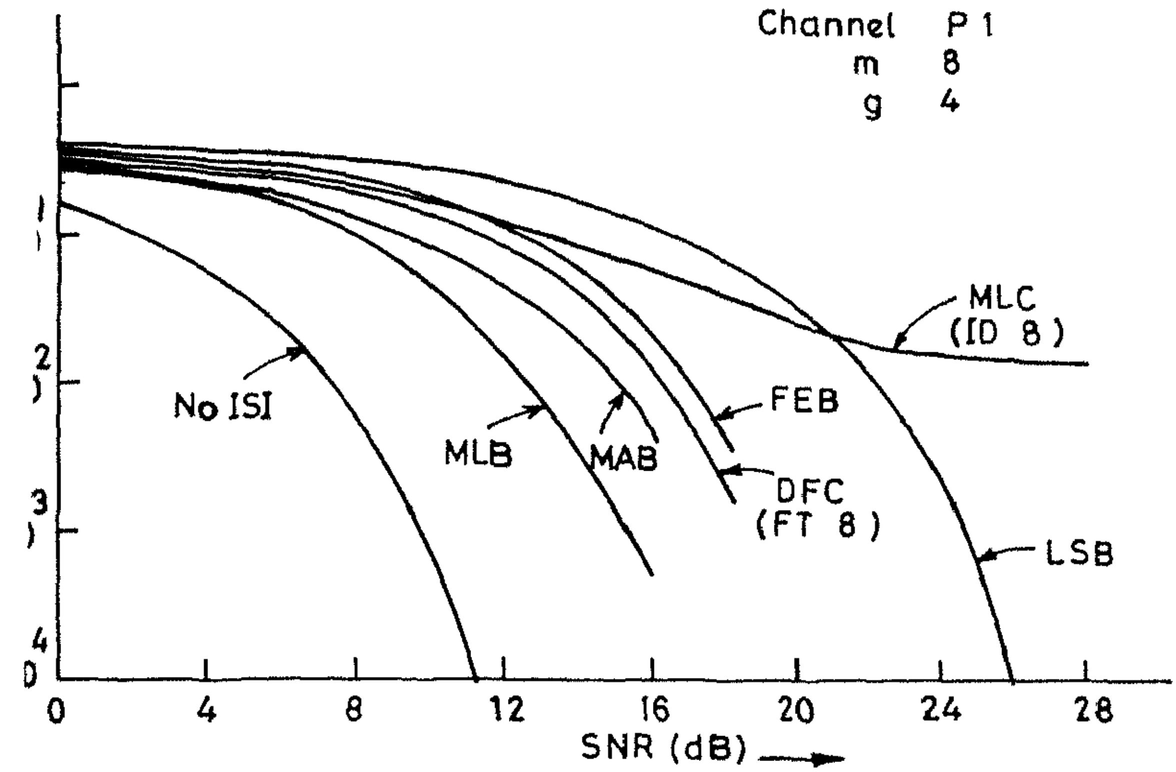


FIG 8 6 Comparison of performance of various DR scheme



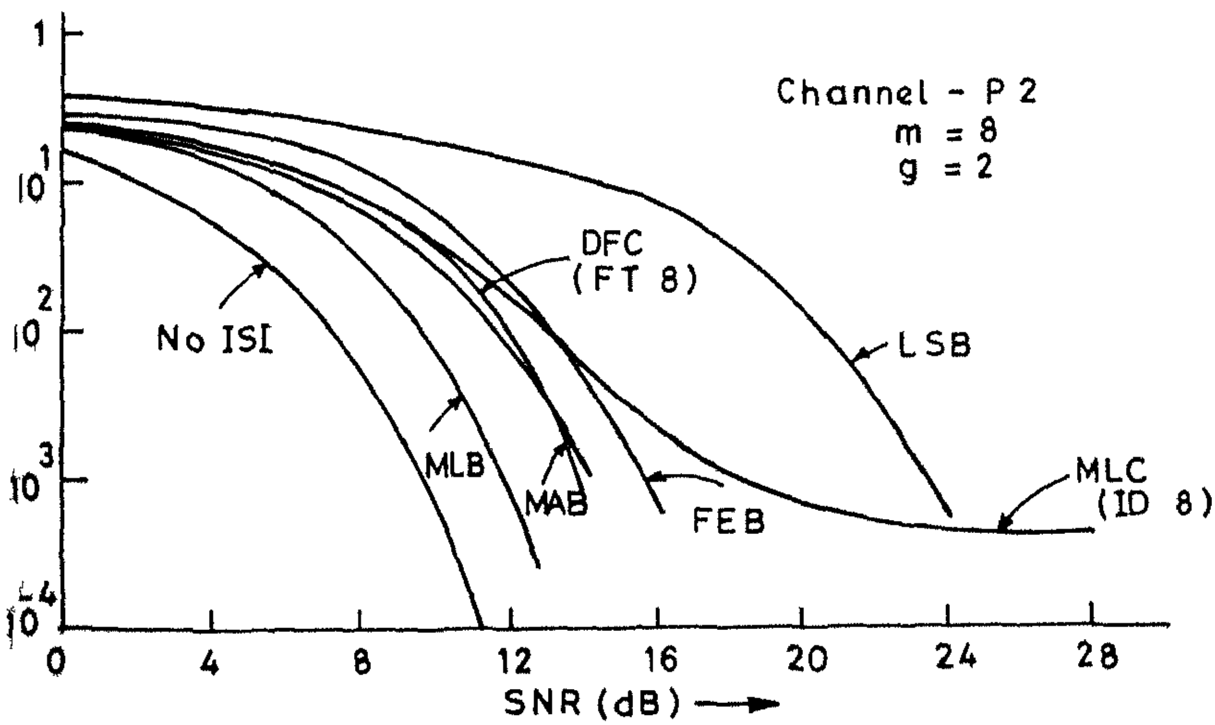
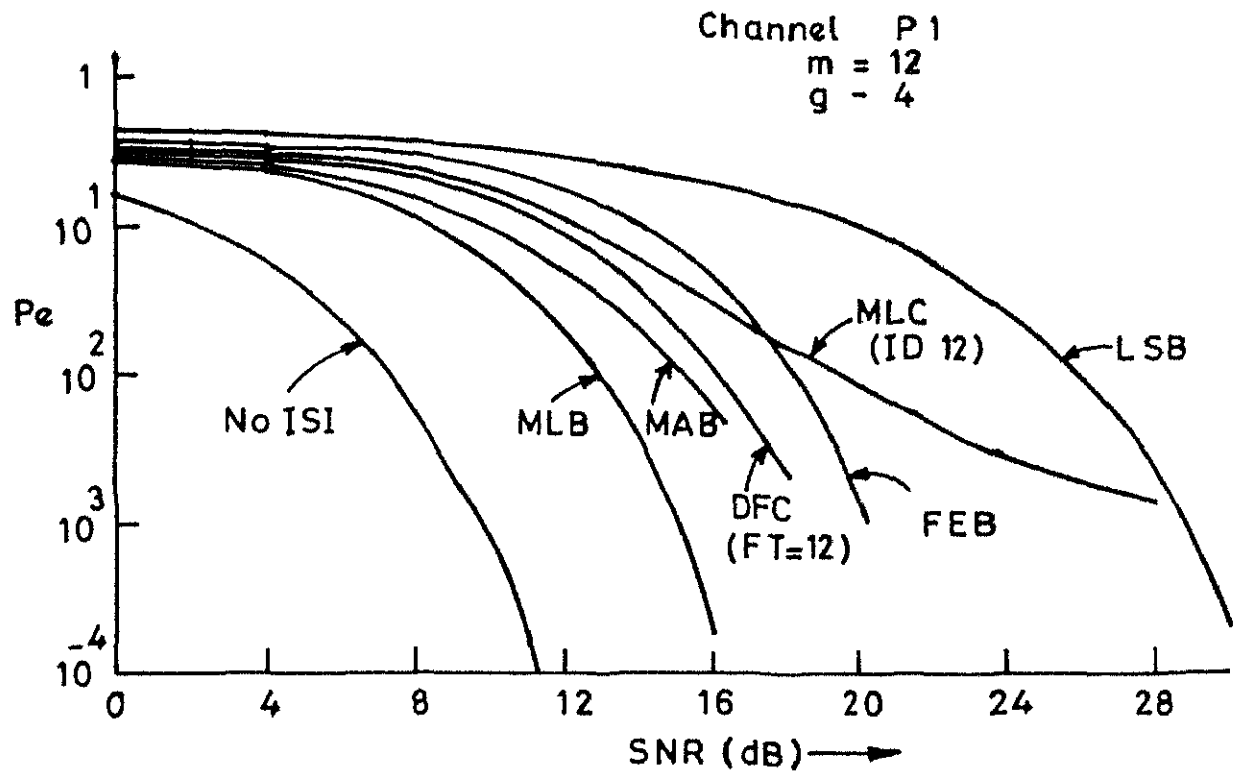


FIG 8 7 Comparison of performance of various DR schemes



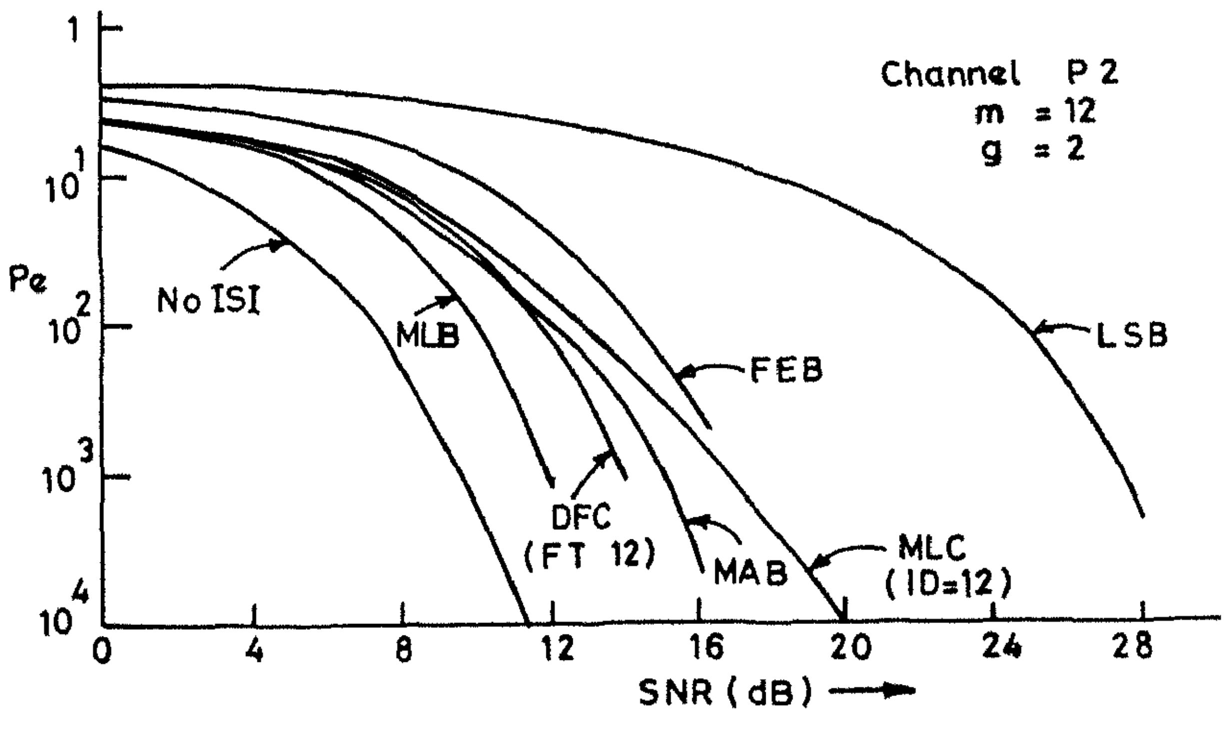
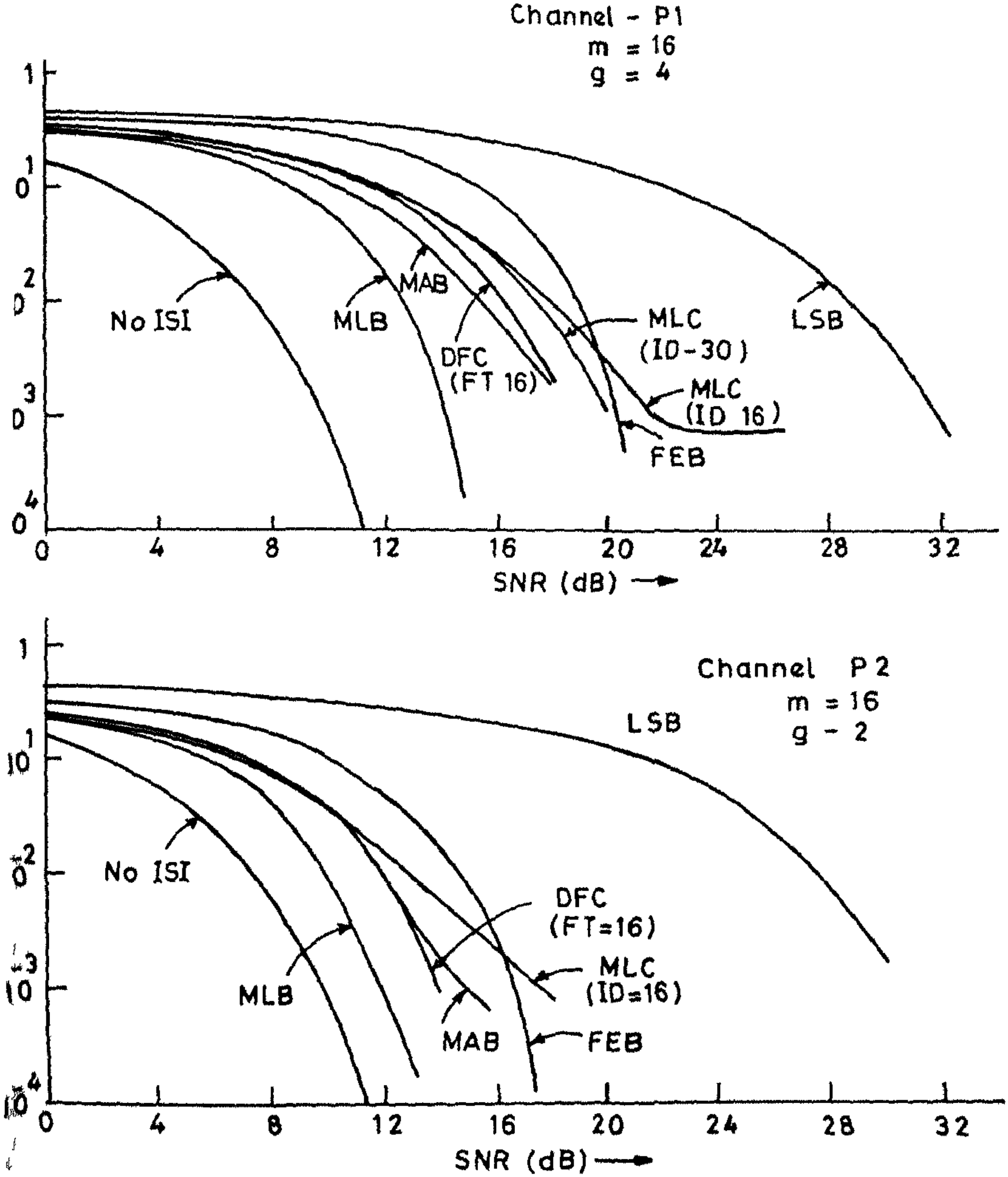


FIG 8 8 Comparison of performance of various DR schemes



IG 8 9 Comparison of performance of various DR schemes

first stage The procedure is continued till all the transmitted data are decided The value of ID i m it oned in the figures The decision feedback scheme for C DT (DFC) is based on the equations given in [7] The number of forward taps used is den ted by FT in the figures For th ake of co pari on error probability curve for channels with no ISI is lso shown in the figures

The Figures 8 1 to 8 9 cover a sufficiently wide rang of block lengths and channel responses The figures indicate that the performance of MLB is better than MLC when ID is equal to the block length In the course of the simulation studies we have observed that for obtaining good performance at medium and large SNR values ID must be large in the case of MLC In Figure 8 10 we have compared the performance of MLB and MLC on th maximum distortion channel $\frac{1}{\sqrt{5}}$ (1 1 1 1 1) Figure 8 10 shows that MLC performance saturates if ID is not la g

From the Figur s 8 1 - 8 9 we also observe that MAB performs well compared to DFC when the block length is equal to the number of forward taps However it must be kept in mind that MAB requir s more memory compar d to DFC It may b recalled that the computational complexity of both he schemes are more or less same FEB how ver does not p rform as well as MAB One reason is that FEB assumes no knowledge of the noise variance One attractive feature of FEB is that it can handle illconditioned channels more efficiently compared to the Trench algorithm

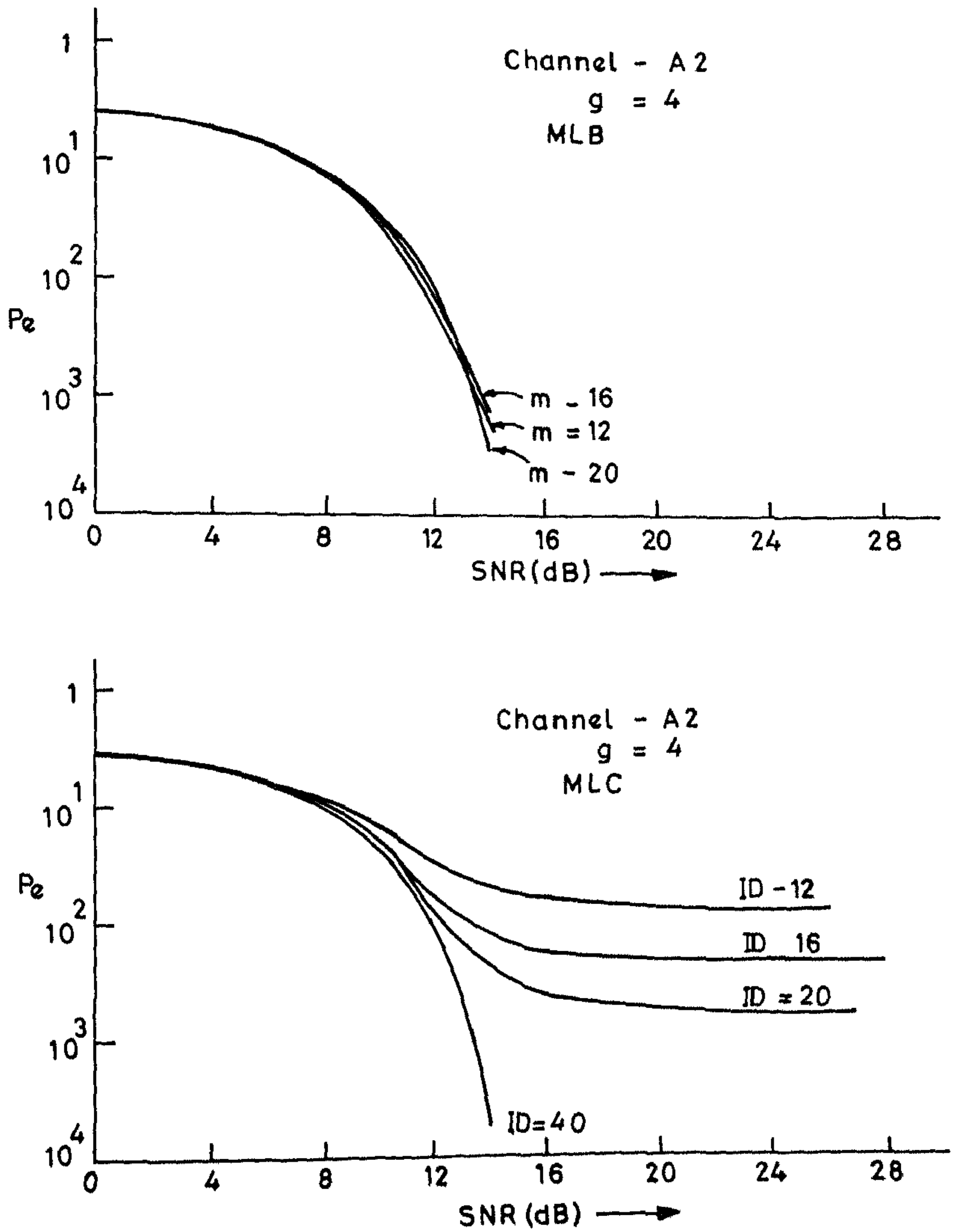


FIG 8 10 Comparison of MLB and MLC performace

based MAB in which illconditioned ch nnels can j ve r se to large values of the tap coefficien s

It is interesting to note that in the case of C1 (see Table 8 2) there is essentially no differ nc between MLC and MLB for block length of 8 12 16 In fac CC performance is also very close to MLB One reason for t is perhaps is that d_{min} for this channel is 1 93 Whereas for the other channels it i much less than 2 (see Table 8 2) In order to check the validity of this claim in Figure 8 11 we have plotted the performance curves fo the example given in [33] For this channel d_{min} is 2 Performance curves indicate that the loss in performance due to CCB compared to MLB is not ignificant This observation however needs more inv stigation

Figure 8 12 shows the effect of block length on the performance of CCB and SCB in the case of channels C1 and P1 Although the performance in the case of channel C1 (d_{min}=1 93) is better compared to that with P1 (d_{min} = 1 13) the perfor mance depends strongly on block l ngth

8 3 1 The Effect of Increased ISI Among the Block Elements

So far we have considered BDT in which the effective data rate over the channel is less than the source data rate due to g zeros following each block W now consider the situation in which the data values of a block are transmitt d at a higher

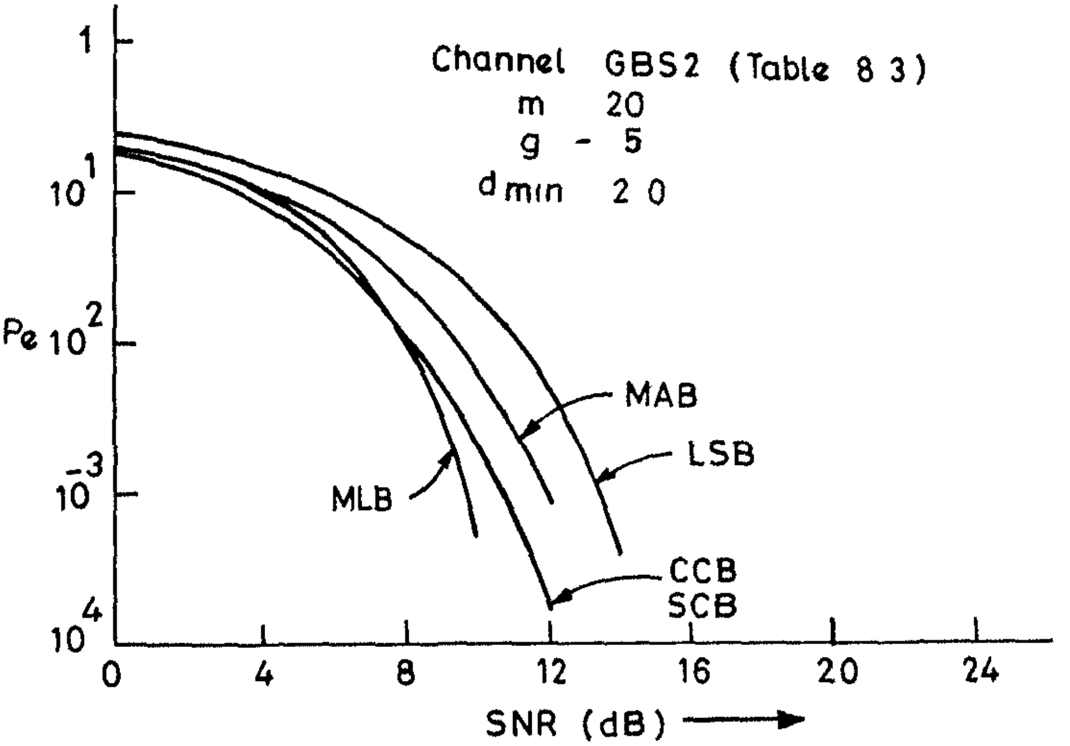
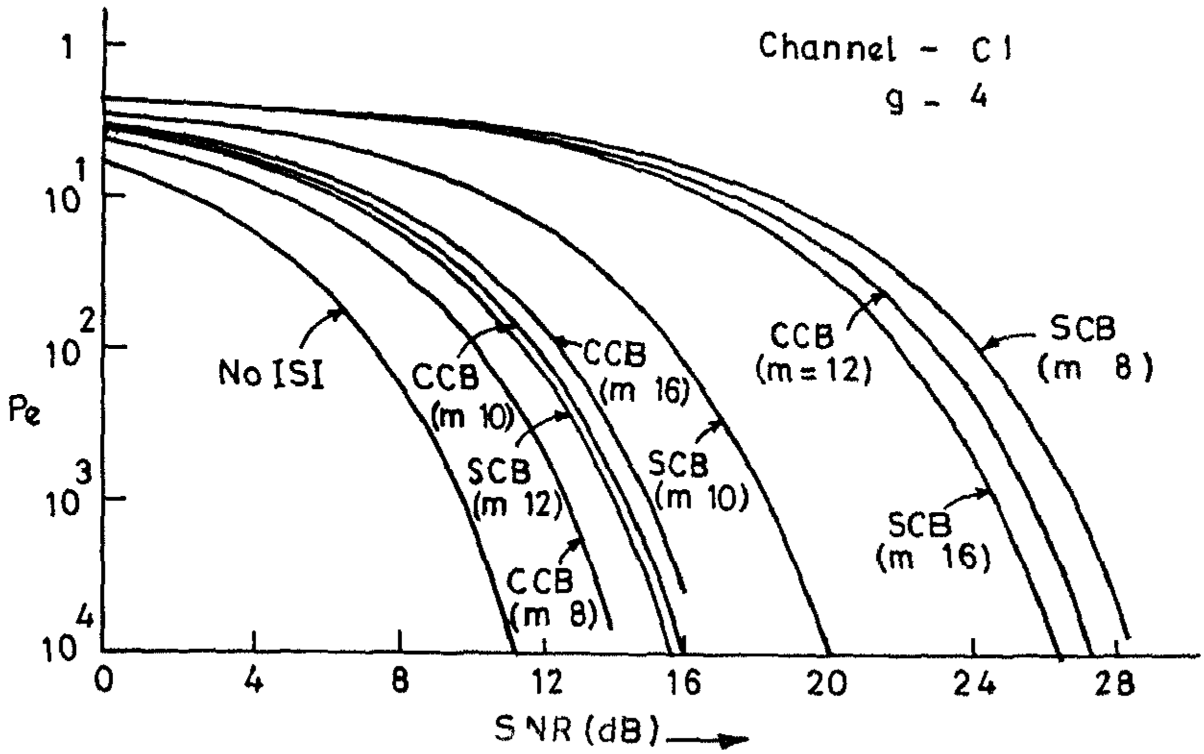


FIG 8 11 MMD Channel performance



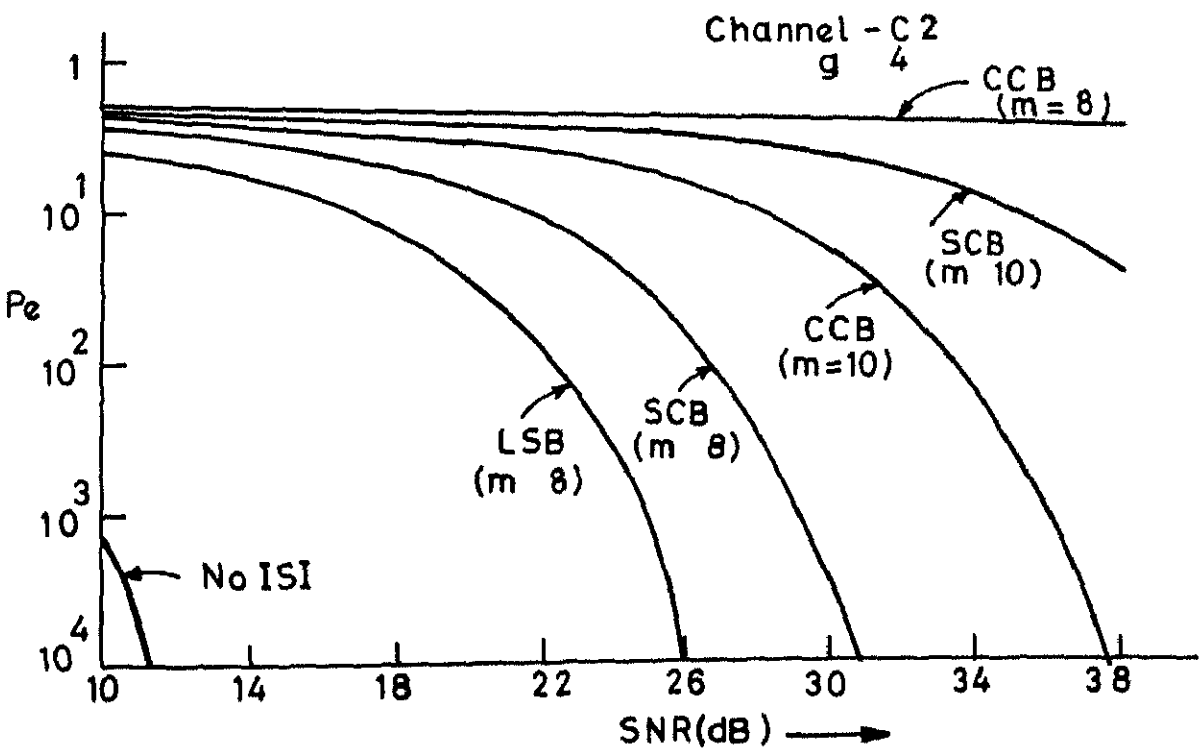


FIG 8 12 Effect of m on the performance of CCB and SCB

rate so that there is no reduction in th d a r t over the channel However this results in increased ISI among the block members The effect of incr asing the data rate on the performance is the topic of this section

First we consider maximum distortion channels In Figure 8 13 we have shown the effect of increasing ISI when MLB is employed For the sake of comparison we have also plotted MLC performance for ID equal to the block length Note that the MLC curve for ID = 16 aturates aft 14 dB

Figure 8 14 shows the perform nce curves in the ca e f DFC and MAB In this case also we obs rve that there is improvement in the performance of MAB compared to DFC

As another example we considered a telephone channel given in [33] In Table 8 3 we have given the equivalent

Table 8 3 Channel sampled response coefficients

Channel	Symbol duration (in nano-seconds)	Channel par meters	d_{min}
GBS1	3	013 - 067 074 403 910	2 0
GBS2	2 5	01 044 - 094 031 561 821	2 0
GBS3	2	017 044 - 086 272 676 678	1 719

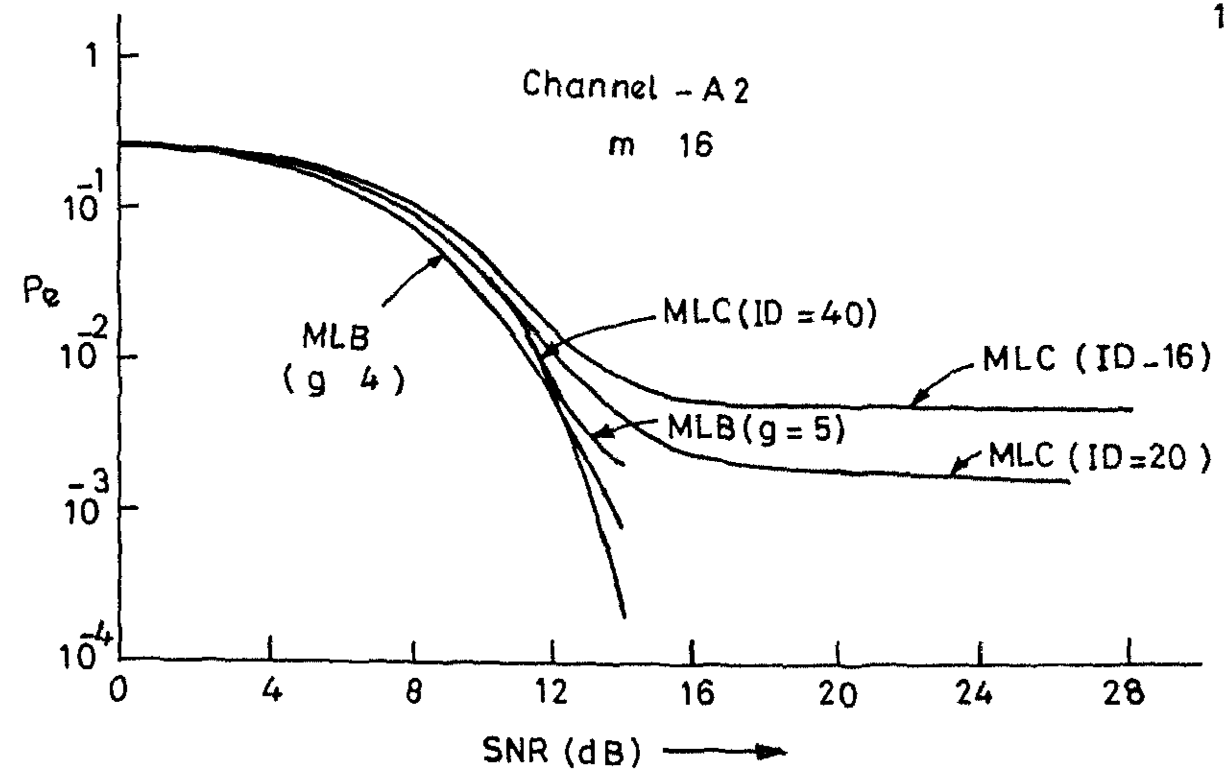


FIG 8 13 Effect of increased ISI on the performance of MLB

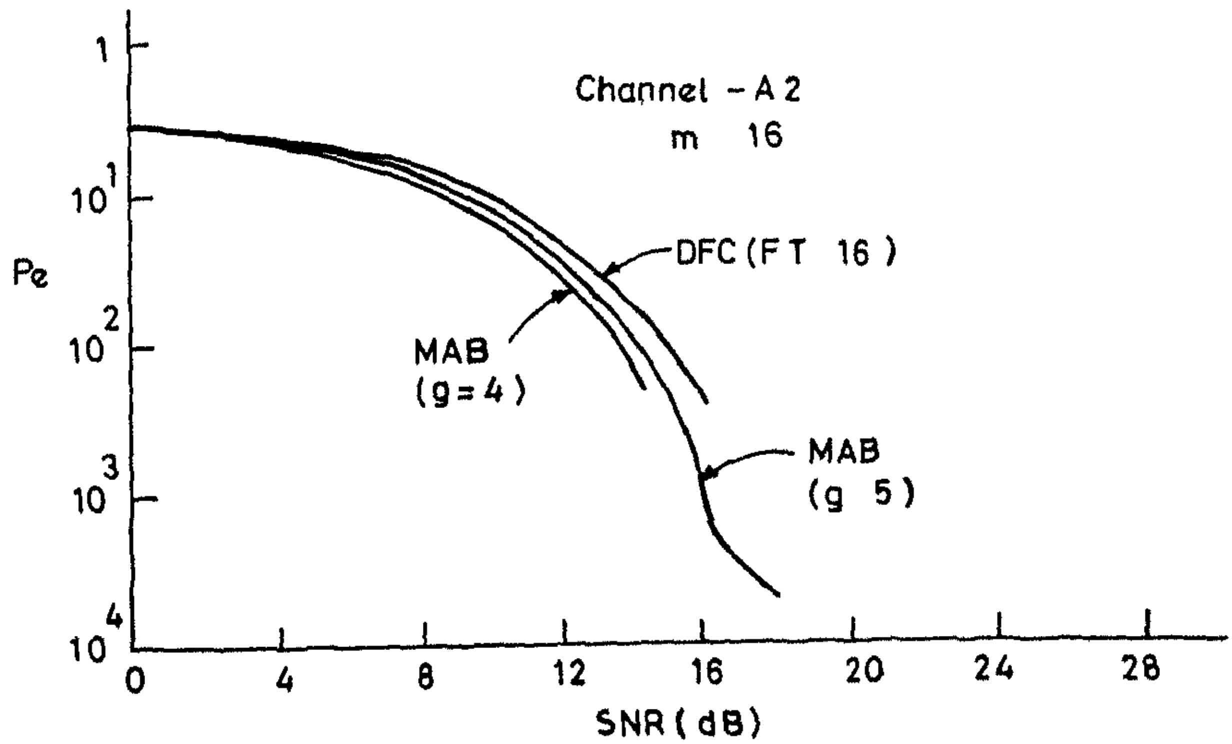


FIG 8 14 Effect of increased ISI on the performance of MAB

discrete time response values for various data ra es The performance curv s are shown in Figures 8 1 J 8 16 In this case however BDT does not seem to enjoy noticeable advantage over CSDI

Summarizing w evaluated th performance of various date recovery schemes using simulation It may be noted that c rc must be exercised while making observations or while drawing conclusions regarding a particular sch nc bas a on simulation results Thus with limited numb r of examples and in the absence of any theoretical justification it is difficult to make any definite conclusion regarding the upcriority of one schem over the other However from our limited simulation eyperiments on typical channels it app ars that by choosing moderate block size and with incr ascd data ra e over the chann l LB can be effectively implemented at a reasonable computational and storage conple co p red to MLC which requires large amount of storage as ID >> 5g Similarly we have obs rved that when the value of d_{min} is close to 2 CCB or SCB works w ll compared to MLB with relatively small amount of SNR loss MAB also seems to be effective compared to DFC although it requires more storage Moreover it limits error propagation to a block length in the worst situation - a feature absent in the case of DFC

CHAPTER 9

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this thesis we have investigated the problem of recovering data in block data transmission systems We have developed several data recovery schemes and compared their performance with DR schemes available for continuous serial data transmission systems The data recovery problem has been viewed as a finite deconvolution problem in the presence of noise Three deconvolution methods have been studied in detail They are maximum likelihood decision feedback and least squares deconvolution methods The DR schemes developed on the basis of these deconvolution methods may be viewed as counterparts of the maximum likelihood decision feedback and linear DR schemes in the context of CSDT

At this stage it may be appropriate to note the following difference between BDT and CSDT systems Due to practical considerations in CSDT systems truncated number of observations are used for making decision regarding a transmitted data symbol although in theory we must use the entire observation values Thus the number of observations used for making decision determines the performance of a given DR scheme In BDT we control the data rate and data transmission format so as to achieve better trade-off between error performance and computational complexity

Sali nt features o the study carried out in this thesis and sugges ions for further research ar given in th following paragraphs

The maximum likelihood recursiv d c nvolution al rithm developed in Chapter 3 is simpl to understand compared to the conventiona Viterbi algorithm approach which m kes use of Bellman s principle of optimality and state r presentation The ML-RDA developed u ing Mit en s l m a provi les mor insight into the data recovery problem compar d to th Viterbi algorithm application The computational compl xity and storage requirements of ML RDA grows linearly with the block length and exponentially with he channel memory Simulation studie on some typic l channels using L RD for BDT and CoDT systems show that ML-RDA for CoDT system requires more storage and computational complexity compared to ML-RDA for BDT systems in order to achieve same error performance

W have further shown that in the case of binary dat it is possible to reduce both storage and computational requirements of ML RDA by using results from the theo y of Boolean and pseudo-Boolean functions Development of similar methods for multil vel data using some of he t hniqu s given in [72] for dealing with functions of discr te variabl s i an interesting area for further research

We have also shown that the method of minimizing pseudo Boolean functions provides a goo framework for developing ML-decoding sch mes for binary error correcting block codes Examples given in Ch pter 3 illustrate the usefuln ss of th method described We however have not studied this aspect in detail The technique d veloped in this thesis can be extended to nonbinary codes al o A detailed study of codes over other fields using some of the r sults available in [72] is expected to prove usefu in a better understanding of ML decoding of linear as well as nonlinear block codes

The decision feedback schemes developed in Chapter 4 are computationally more attractive in s tuations where ML-RDA becomes uneconomical Unlike the conventional methods of deriving decision feedback data recovery schemes w have motivated the idea of decision f edback via Gaussian elimination method We have shown that th modified Austin approach can also be viewed as Gaussian elimination method with a nonlinear decision device

The DF-DR schemes for BDT systems restrict error propagation to atmost a block length whereas in the cas of DF-DP schemes for CSDT systems error propagation can extend to much longer durations However DR sch mes f r BDT systems require more storage compared to DR schemes for CSDT systems for achiev ng same probability of error performance

The DR scheme studied in Chapter 5 have the lea t computational and storage requirements compared to t cse studied in Chapters 3 and 4 However thi i only at t e expense of probability of error performarce However simulation studies show that in the case of a good number of channels DR schemes based on k Circulant completions per orms well compared to DR schemes based on LS-deconvolution We have shown that k Circulant completions of th channel convolution matrix provides a more gen ral etting for studying deconvolution problems using di cr te Fourier transform A detailed study of optimum choice of k-Circulant completions according to some performance criteria needs further investigation

In Chapter 6 we have showed hat deconvolution using DFT and least squares deconvolution can be viewed as particular cases of square completions of the channel convolution atrix

Detailed anal/ is of he Steepest d scent algorithm in Chapter 7 shows that in the case of some examples good estimates of the convergence parameter can be obtained using l-Circulant completions This analysis clearly explains the observation made by others that DFT domain fixed tep steepest descent algorithm converges faster han the sa ple domain chemes Moreover w have shown that such estimat s can be used in the sample domain o achiev good convergence

The sımulatıon results gıvcn ın Chaptcr 8 cov ~ a ʌı J range of blɔck lengths and typıc l channel exa pl s It ıs ınterestıng to note that the effec of ıncr as^d ISI among the symbols of a block due to ıncrcased data rate ov r the channcl on the performance ıs not sıgnıfıcant Sımulatıon studıes also ındıcate that vıth approprıate ch ıcc of block lengths BDT provıdes better tr dc-off b twcen rror rformance and proccssıng c mplexıty compar d to th av ıl blc DR schemes for CSDT syste ıs Howev r as ımulatıon results arc always bɐsed on a few examples n defınıte claıms can be made regardıng the superıorı y of BDT ov r CSDT

The maxımum mınımum dıstance and maxımum dıst nce ch_nnels obtaın d ın Chapter 8 show th t scme of the good channcls are partıal response chɐnnels How ver ın the absence of any clear defınıtıon of partıal response channel [94] ıt ıs dıffıcult to show the connectıon between partıal rcsponse nd MMD or MD channels Thıs asp ct needs further ınvestıgatıon

Another aspcct of the data r covery problem ın vhıch we could not achıeve much success ıs ın obtaınıng gɔ d bounds for the crror p rform nce of max mum lıkelıhɔod and d cısıon f dback sch mes The dıffıculty seems tɔ be du to the nonlınear nɐture of the DR schemes However w stıll f el that ıt must b possıble to obtaın useful bounds for probabılıty of error whıch ın thıs case wıll be functıons of block length and channel

memory ihi aspect requir s further investigation Sone work in this direction has b en carried out by Klein and Wolf [54]

In his thesis we have not considered the problem of adapting the parameters of th data recovery schem s to the channel variations Depending on the nature of the channel variation the design of th dat r covery schemes will change The problem is interesting due to th presence of noise and our limitation in recovering data witnout errors Further investigation is needed to understand various aspects of this problem

APPENDIX A

MITTEN S LEMMA

In this appendix we state and prove Mitten lemma [55]

Mitten s Lemma Let $X = (x)$ be an arbitrary set and let $Y_i(x) = (y_i)$ $i = 1\ 2\ \ \ n$ be finite sequence o arbitrary sets whose definitions may depend on $x \varepsilon X$ Let $g_i(x\ y_i)$ $i - 1\ 2\ \ \ n$ be a sequence of real valued functions for which $g_i^*(x) = \min[g_i(x\ y_i) | y_i \varepsilon Y_i(x)]$ exists for $i = 1\ 2\ \ \ n$ and all $x \varepsilon X$ Let $f(x)$ be an arbitrary function and let $g(x\ y_1\ \ \ y_n) - h[f(x)\ g_1(x\ y_1)\ \ \ g_n(x\ y_n)]$ be a real valued function for which $g^* = \min[g(x_1\ y_1\ \ \ y_n) |$ $x \varepsilon X\ y_i \varepsilon Y_i(x)\ i = 1\ 2\ \ \ n]$ ists and $h^*(x) = h[f(x)$ $g_1^*(x)\ \ \ g_n^*(x)]$ exist for all $x \varepsilon X$ If $h\ (x) \leq g(x_1\ y_1\ \ \ y_n)$ for very $x \varepsilon X$ and $y_i \varepsilon Y_i(x)$ $i = 1\ 2\ \ \ n$ then $g\ = \min[h^*(x) | x \varepsilon X] = h$

Proof The lemma can b proved by the method of contradiction For this purpose assum that there exists $x\ \varepsilon X$ and $y_i \varepsilon Y_i(x\ )$ $i = 1\ 2\ \ \ n$ such that $g(x\ \ y_1\ \ \ y_n) < h$

But from the definition of minimum $h^* \leq h^*(x\ )$ and $h\ (x\ ) \leq g(x\ \ y_1\ \ \ y_n)$ by the assumed property of $h^*(x)$ Therefore

$$g(x\ \ y_1\ \ \ y_n) < h^* \leq h^*(x\ ) < g(x\ \ y_1\ \ \ y_n)$$

which is a contradiction

APP I DIX B

k CIRCUL IT IATRICE

The purpose of this appen ix i to list a few properties of k-Circulant matrices [81] [82]

The k-Circulant matrices con idered in our study have the following structure

$$C_k = \begin{bmatrix} C_o & kC_{n-1} & kC_{n-2} & & kC_1 \\ C_1 & C_o & kC_{n-1} & & kC_2 \\ C_2 & C_1 & C_o & & \\ & & & & \\ C_{n-1} & C_{n-2} & C_{n-3} & & C_o \end{bmatrix}$$

where $|k| = 1$ When $k = 1$ C_1 is a Circulant matrix $k = -1$ gives skew Circulant matrix Skew Circulants ar also known as negacyclic matrices [82]

It is convenient to study k-Circulant with the help of the simple k-Circulan matrix

$$\eta_k = \begin{bmatrix} 0 & kI_1 \\ & \\ I_{n-1} & 0_{n-1\ n-1} \end{bmatrix}$$

A matrix A is said to be k Circulant iff

$$A\eta_k = \eta_k A$$

in terms of η_k C_k can be expressed as

$$C_k = C_o I_n + C_1\eta_k + C_2\eta_k^2 + \quad C_{n-1}\eta_k^{r-1} \qquad (B\ 1)$$

S nce every k Circulant matrix commutes with η_k it is sufficient to know the eigenvalu s and eig nvectors of η_k in order to characterise and analys Circulants

The eigenvalu s of η_k are the nth root of k for the characteristic equation of η_k is given by

$$\lambda^n - k = 0$$

Let p be an nth root of k Th n $p\omega^i$ $i = 0\ 1 \quad n-1$ where $\omega = e^{j2\pi/n}$ represents n roots of $\lambda^n - k = 0$

We now show that th v ctor

$$x = (1\ x^{-1}\ x^{-2} \quad x^{-(n-1)})^T$$ wh re x is an nth root of k is a eigenvector of η_k

A simple calculation shows that

$$\eta_k\ x = \begin{bmatrix} k\ x^{-(n-1)} \\ 1 \\ x^{-1} \\ x^{-2} \\ x^{-(n-2)} \end{bmatrix} = x\ \underline{x}$$

Thus n roots of k generate n eigenvectors of η_k Let F_k^{-1} be a matrix with the n eigenv ctors of η_k as columns It has the following form

$$F_k^{-1} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 \\ p^{-1} & (p\omega)^{-1} & (p\omega^{n-1})^{-1} \\ p^{-2} & (p\omega)^{-2} & (p\omega^{n-1})^{-2} \\ & & \\ p^{-(n-1)} & (p\omega)^{-(n-1)} & (p\omega^{n-1})^{-(n-1)} \end{bmatrix} \quad (B\ 2)$$

Note that $\frac{1}{\sqrt{n}}$ is just a normalization factor used for making $F_k^{-1}\Gamma_k = I_n$

Using B 2 it can b shown that

$$F_k \eta_k F_k^{-1} = \Lambda_k \quad (B\ 3)$$

where Λ_k is a diagonal matrix with $p\omega^i$ i = 0 1 n-1 as d agonal elements

Wh n k = 1 $F_k = F_1 = F$ the DFT matrix is obtained In virtue of this we shall refer to F_k as κ-DFT m trix In t rms of F we can write F_k^{-1} a

$$F_k^{-1} = D^{-1}F^{-1} \quad (B\ 4)$$

where D is a diagonal matrix with p^i i = 0 1 n-1 as diagonal entries Using (B 4) in (B 3) w obtain

$$F\, D\, \eta_k\, D^{-1} F^{-1} = \Lambda_k \qquad (B\ 5)$$

If C_1 is a Circulant matrix we know that

$$F\, C_1\, F^{-1} = \Lambda_1 \qquad (B\ 6)$$

where Λ_1 is a diagonal matrix with eigenvalues of C_1 as diagonal entries

Comparing (B 6) and (B 5) we see that $D\, \eta_k\, D^{-1}$ represents a Circulant matrix

Post and pre multiplying B 1 by F_k^{-1} and F_k respectively and using (B 3) we obtain

$$F_k\, C_k\, F_k^{-1} = C_o I + C_1 \Lambda_k + C_2 \Lambda_k^2 + \quad C_{n-1} \Lambda_k^{n-1} \qquad (B\ 7)$$

From (B 7) we see that if λ_i $i = 0\ 1 \quad n\ 1$ are the eigenvalues of C_k

$$\lambda_i = \sum_{j=0}^{n-1} C_j (p\omega^i)^j \qquad (B\ 8)$$

(B 8) can also be rewritten as

$$\lambda_i = \sum_{j=0}^{n-1} (C_j p^j)\, \omega^{ij} \qquad (B\ 9)$$

(B 9) exhibits an interesting aspect eigenvalues of C_k can be determined by finding the eigenvalues of a Circulant matrix having first column

$$(C_o\ C_1 p\ C_2 p^2 \qquad C_{n-1} p^{n-1})^T \qquad (B\ 10)$$

This can also be seen directly u ing (B 7) Using (B 4) in (B 7) we get

$$FDC_kD^{-1}\Gamma^{-1} = \Lambda = \mathrm{diag}(\lambda_o \lambda_1 \quad \lambda_{n\ 1}) \qquad (B\ 11)$$

Therefore DC_kD^{-1} is a Circulant matrix By little calculation we can see that the first column of this n trix is given by (B 10)

Finally consider inverse of a k-Circulant riting (B 3) in terms of η_k^{-1} whenever $\eta_k^{\ 1}$ exists le get

$$\eta_k^{-1} = \Gamma_k^{\ 1} \Lambda_k^{-1} \quad k$$

or $$F_k\eta_k^{-1}F_k^{-1} = \Lambda_k^{-1}$$

which shows that inverse of a k Circulant whenever it exists is also a k-Circulant

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